Chapter 1 . Angular Momentum Algebra

In this chapter first we will review some of the basics of angular momentum algebra, followed by a discussion of addition of angular momenta and the Clebsch – Gordon series. Finally we will discuss Wigner - Eckart theorem.

Revision of Basics: (i) Basic Algebra

Angular momentum operator $\vec{J}$ is a Hermitian vector operator defined as

$$\vec{J} = J_x \hat{i} + J_y \hat{j} + J_z \hat{k} \quad \text{—(1)}$$

where J_x, J_y, J_z are its three Cartesian components. Hermiticity condition

$$\vec{J} = \vec{J}^\dagger \quad \text{—(2)}$$

clearly implies that the individual components are Hermitian

$$\left. J_x^\dagger = J_x \,,\; J_y^\dagger = J_y \,,\; J_z^\dagger = J_z \right\} \text{—(3)}$$

Additionally, the three components of the angular momentum must satisfy the commutation relations (2)

$$\left.\begin{aligned} [J_x, J_y] &= i\hbar J_z \\ [J_y, J_z] &= i\hbar J_x \\ [J_z, J_x] &= i\hbar J_y \end{aligned}\right\} \quad — (3)$$

which can be written in the compact form

$$\boxed{[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k} \quad — (4)$$

~~Operating on J^2~~ Using these commutation relations, we can show that the operator $J^2 = J_x^2 + J_y^2 + J_z^2$ commutes with each individual component

$$[J^2, J_i] = 0 \quad — (5)$$

Eq.(5) implies that J^2 and J_i are simultaneously diagonalizable, i.e., they have common eigenvectors. But, because different components J_i don't commute, we cannot obtain their shared eigenvectors. Thus, by convention one obtains common eigenvectors of J^2 and J_z labeled $|jm\rangle$, which satisfy

$$\left.\begin{aligned} J^2|jm\rangle &= j(j+1)\hbar^2|jm\rangle \\ J_z|jm\rangle &= m\hbar|jm\rangle \end{aligned}\right\} \quad (6)$$

where $-j \leq m \leq j$, and $|jm\rangle$ form an orthonormal set of vectors

$$\boxed{\langle j'm'|jm\rangle = \delta_{jj'}\delta_{mm'}} \quad (7)$$

One can also show that allowed values of j satisfy

$$\boxed{j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots} \quad (8)$$

and that successive m values for a given j get incremented by one, i.e. $m' = m \pm \frac{1}{2}$. Combining this with the fact that $-j \leq m \leq j$, we conclude that for each value of j, there are $2j+1$ allowed m values $-j, -j+1, -j+2, \ldots (j-1), j$.

To simplify calculations it is useful to define ladder operators

$$J_\pm = J_x \pm i J_y \quad \text{—— (9)}$$

which are clearly non-Hermitian and satisfy

$$J_\pm^\dagger = J_\mp \quad \text{—— (10)}$$

one can write

$$J^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2 \quad \text{—— (11)}$$

Using the commutation relations (Eq. 4) one can show that operation $J_\pm$ on $|jm\rangle$ leads to a ket $|j m'\rangle$, with $m' = m \pm 1$.

To be precise

$$\boxed{\begin{aligned} J_+ |j m\rangle &= \sqrt{(j-m)(j+m+1)}\, \hbar\, |j\, m+1\rangle \\ J_- |j m\rangle &= \sqrt{(j+m)(j-m+1)}\, \hbar\, |j\, m-1\rangle \end{aligned}} \quad \text{—— (12)}$$

Note that J_i operators are general angular momentum operators, which in practice could refer to orbital angular momenta L_i, spin angular momenta S_i, their sum $L_i + S_i$ or their other combinations, as long as they satisfy commutation relations of Eq. (a).

(ii) Generators of Rotation:

Active view of rotating with coordinate system fixed

Suppose we rotate the ~~coordinate~~ the position vector $\vec{r}$ ~~system~~ by an angle ϕ, about an axis along the direction $\hat{n}$, then the Hamiltonian ~~for all~~ $\mathcal{H}$ of that system will transform according to ~~mode~~ rule

$$H_R = U_R H U_R^{+} \quad \text{——} (13)$$

and a given ket $|\alpha\rangle$ will transform to

$$|\alpha\rangle_R = U_R |\alpha\rangle, \quad \text{——} (14)$$

where H_R and $|\alpha\rangle_R$ are quantities with respect to the rotated coordinate system and

$$U_R = e^{-\frac{i}{\hbar}\hat{n}\cdot\vec{J}\phi} \quad \text{——} (15)$$

where $\vec{J} = J_x\hat{i} + J_y\hat{j} + J_z\hat{k}$.

~~and~~ U_R is called the rotation operator, and $\vec{J}$ is the generator of rotation in the 3D ~~coord space~~ Euclidean space. Note, we have taken an active view of rotation

(iii) Representation of Rotation Operator:

Representation of the rotation operator with respect to the basis $\{|jm\rangle; m=-j,\dots+j\}$ are called rotation matrices

$$U_R|jm\rangle = \sum_{m'} |jm'\rangle\langle jm'|U_R|jm\rangle$$

(we used the relation $\sum_{m'} |jm'\rangle\langle jm'| = I$)

defining

$$D^{(j)}_{m'm}(R) = \langle jm'|U_R|jm\rangle$$
$$= \langle jm'|e^{-\frac{i}{\hbar}\vec{J}\cdot\vec{\phi}}|jm\rangle \quad (16)$$

so that

$$\boxed{U_R|jm\rangle = \sum_{m'} D^{(j)}_{m'm}(R)\,|jm'\rangle} \quad (17)$$

thus $D^{(j)}_{m'm}(R)$ is nothing but the ~~ro~~ representation U_R with respect to the given basis. $D^{(j)}_{m'm}(R)$ are called ~~ro~~ rotation matrices and are difficult to compute except for simple cases. For example, for a rotation about z axis, i.e.,

$$U_R = e^{-\frac{i}{\hbar} J_z \phi}$$

$$\boxed{D^{(j)}_{m'm}(R) = e^{-\frac{i}{\hbar} m\phi} \delta_{mm'}}$$

The calculation of $D^{(j)}_{m'm}(R)$ for a general rotation can be simplified if we use the concept of Euler rotations from classical mechanics. We recall that a general 3D rotation can be written in a simplified form in terms of three rotations by angles α, β, and γ, in counter-clockwise directions ~~about~~ as described next. The first rotation by angle α is about the original z axis, second rotation by angle β is about the new y axis, and the third rotation by angle γ is about the new z axis. If the original axis are (x, y, z), intermediat intermediate ones are (x'', y'', z'') and the final ones are (x', y', z'), then clearly

$$U_R = e^{-\frac{i}{\hbar}\gamma \hat{z}'\cdot\bar{J}}\, e^{-\frac{i}{\hbar}\beta \hat{y}''\cdot\bar{J}}\, e^{-\frac{i}{\hbar}\alpha \hat{z}\cdot\bar{J}} \quad (18)$$

using a mathematical trick, one can transform Eq(18) into a form involving rotation strictly about the original axes. This involves realization

$$e^{-\frac{i}{\hbar}\beta \hat{y}''\cdot\bar{J}} = e^{-\frac{i}{\hbar}\alpha \hat{z}\cdot\bar{J}}\, e^{-\frac{i}{\hbar}\beta \hat{y}\cdot\bar{J}}\, e^{\frac{i}{\hbar}\alpha \hat{z}\cdot\bar{J}} \quad (19)$$

and
$$e^{-\frac{i}{\hbar}\gamma \hat{z}'\cdot\bar{J}} = e^{-\frac{i}{\hbar}\beta \hat{y}''\cdot\bar{J}}\, e^{-\frac{i}{\hbar}\gamma \hat{z}\cdot J_z}\, e^{\frac{i}{\hbar}\beta \hat{y}''\cdot\bar{J}} \quad (20)$$

On substituting (19) and (20) into (18), we obtain the simplified form containing rotations about the original axis

$$U_R = e^{-\frac{i}{\hbar}\alpha \hat{z}\cdot\bar{J}}\, e^{-\frac{i}{\hbar}\beta \hat{y}\cdot\bar{J}}\, e^{-\frac{i}{\hbar}\gamma \hat{z}\cdot\bar{J}}$$
$$= e^{-\frac{i}{\hbar}\alpha J_z}\, e^{-\frac{i}{\hbar}\beta J_y}\, e^{-\frac{i}{\hbar}\gamma J_z} \quad (21)$$

This leads to a much simpler expression for the rotation matrices.

$$D^{(j)}_{m',m}(\alpha,\beta,\gamma) = \langle jm'| e^{-\frac{i}{\hbar}\alpha J_z} e^{-\frac{i}{\hbar}\beta J_y} e^{-\frac{i}{\hbar}\gamma J_z} |jm\rangle$$

$$= e^{-i\alpha m'} e^{-i\gamma m} \langle jm'| e^{-\frac{i}{\hbar}\beta J_y} |jm\rangle$$

$$= e^{-i\alpha m' - i\gamma m} \langle jm'| e^{-\frac{\beta}{2\hbar}(J_+ - J_-)} |jm\rangle \quad \text{— (21)}$$

One can also verify the following symmetry properties of rotation matrices

$$\left.\begin{aligned} D^{(j)*}_{m'm}(\alpha,\beta,\gamma) &= D^{j}_{m'm}(-\gamma,-\beta,-\alpha) \\ D^{(j)*}_{m'm}(\alpha,\beta,\gamma) &= (-1)^{m-m'} D^{(j)}_{-m',-m}(\alpha,\beta,\gamma) \end{aligned}\right\} \quad \text{— (22)}$$

Orbital Angular Momentum & Rotation Matrices :

If $j = \ell$ (an integer ≥ 0), then we know

$$\langle \bar{r}|\ell m\rangle = Y_{\ell m}(\theta,\phi) \quad \text{— (23)}$$

Let us explore the influence of a rotation

R on $|lm\rangle$

$$|lm\rangle' = U_R|lm\rangle = \sum_{m'} |lm'\rangle\langle lm'|U_R|lm\rangle$$

or

$$|lm\rangle' = \sum_{m'} D^{(l)}_{m'm}(R)\,|lm'\rangle \quad \text{— (24)}$$

or

$$\langle \vec{r}|lm\rangle' = \sum_{m'} D^{(l)}_{m'm}(R)\langle r|lm'\rangle$$

$$Y_{lm}(\theta',\phi') = \sum_{m'} Y_{lm'}(\theta,\phi)\,D^{l}_{m'm}(R) \quad \text{— (25)}$$

Note that (θ,ϕ) and (θ',ϕ') denote the coordinates of the same point in space w.r.t. the original and rotated axes.

We can invert Eq. (25) using the unitary property of the rotation ~~[illegible]~~ matrices.

$$\boxed{Y_{lm}(\theta,\phi) = \sum_{m'=-l}^{l} D^{*(l)}_{mm'}\,Y_{lm'}(\theta',\phi')} \quad \text{— (26)}$$

for a point on the rotated z axis, i.e., z' axis $\theta'=0$, and using the fact that $Y_{lm}(\theta=0,\phi) = \sqrt{\frac{2l+1}{4\pi}}\,\delta_{m0}$ — (27)

we obtain from Eq. (26)

$$Y_{\ell m}(\beta,\alpha) = \sum_{m'=-\ell}^{\ell} D^{*(\ell)}_{mm'}(R)\sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m'0}$$

$$\Rightarrow \boxed{D^{*(\ell)}_{m0}(R) = \sqrt{\frac{4\pi}{2\ell+1}}\,Y_{\ell m}(\beta,\alpha)} \quad -(28)$$

above we used the fact that a point on the z' axis corresponds to $\theta=\beta$, and $\phi=\alpha$, in the original coordinate system.

Using (25) & (28) we can obtain another interesting result. For the purpose we set $m=0$ in (25)

$$Y_{\ell 0}(\theta',\phi') = \sum_{m'=-\ell}^{\ell} Y_{\ell m'}(\theta,\phi)\,D^{\ell}_{m'0}(R)$$

$$= \sqrt{\frac{4\pi}{2\ell+1}}\sum_{m'=-\ell}^{\ell} Y^{*}_{\ell m'}(\beta,\alpha)\,Y_{\ell m'}(\theta,\phi)$$

but $Y_{\ell 0}(\theta',\phi') = \sqrt{\dfrac{2\ell+1}{4\pi}}\,P_\ell(\cos\theta')$

$$\Rightarrow \boxed{P_\ell(\cos\theta') = \left(\frac{4\pi}{2\ell+1}\right)\sum_{m=-\ell}^{\ell} Y^{*}_{\ell m}(\beta,\alpha)\,Y_{\ell m}(\theta,\phi)} \quad -(29)$$

Eq. (29) is nothing but addition theorem of sperical harmonics.

Direct Sum & Direct Product Spaces:

Suppose we have two vector spaces $\mathcal{E}_1 \{|a_i\rangle, i=1 \ldots n\}$ and $\mathcal{E}_2 \{|b_j\rangle, j=1 \ldots m\}$. Operations of direct sum and direct product lead to larger dimensional subspaces than the individual subspaces, as defined below.

(i) **Direct Sum**: The direct sum space $\mathcal{E}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ is denoted as

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2, \quad (30)$$

and it is a vector space of dimension $l = m+n$, with the ordered basis $\{|a_1\rangle, |a_2\rangle \ldots |a_n\rangle, |b_1\rangle \ldots |b_m\rangle\}$

If $v \in \mathcal{E}_1$ and $u \in \mathcal{E}_2$ with

$$v \equiv \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$u \equiv \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$$

then

$$W = V \oplus U$$

$$W \equiv \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ u_1 \\ \vdots \\ u_m \end{pmatrix} \quad \text{--- (31)}$$

For operators $A : \mathcal{E}_1 \to \mathcal{E}_1$ and $B : \mathcal{E}_2 \to \mathcal{E}_2$

with

$$A \equiv \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad \text{--- (32)}$$

and

$$B \equiv \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mm} \end{pmatrix} \quad \text{--- (33)}$$

Then $C = A \oplus B$ with $C : \mathcal{E} \to \mathcal{E}$

$$C \equiv \begin{pmatrix} a_{11} \cdots a_{1n} & 0 \\ \vdots & \\ a_{n1} \cdots a_{nn} & b_{11} \cdots b_{1m} \\ & \vdots \\ 0 & b_{m1} \cdots b_{mm} \end{pmatrix} \quad \text{--- (34)}$$

(ii) **Direct/Tensor Product:**

The direct (or tensor) product of $\mathcal{E}_1$ and $\mathcal{E}_2$ is denoted as

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \quad \text{— (35)}$$

and is an $n \times m$ dimensional space with the ordered basis

$\{ |a_i\rangle \otimes |b_j\rangle ; \ i = 1, \ldots n, \ j = 1, \ldots, m \}$.

Various notations are used to denote the direct product basis

$$|a_i\rangle \otimes |b_j\rangle = |a_i\rangle |b_j\rangle = |a_i b_j\rangle \quad \text{— (36)}$$

The direct product of two vectors $V \in \mathcal{E}_1$ and $U \in \mathcal{E}_2$

$W = V \otimes U$ can be derived easily

Let

$$V = \sum_{i=1}^{n} v_i |a_i\rangle \equiv \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ v_n \end{pmatrix} \quad \text{— (37)}$$

and

$$u = \sum_{j=1}^{m} u_i |b_j\rangle \quad \text{---} \; (38)$$

$$\equiv \begin{pmatrix} u_1 \\ \vdots \\ \vdots \\ u_m \end{pmatrix} \quad \text{---} \; (38)$$

then

$$W = V \otimes U = \sum_{i=1}^{n} \sum_{j} v_i u_j |a_i b_j\rangle$$

$$= \begin{pmatrix} v_1 u_1 \\ \vdots \\ v_1 u_m \\ v_2 u_1 \\ \vdots \\ v_2 u_m \\ \vdots \\ v_n u_1 \\ \vdots \\ v_n u_m \end{pmatrix} \quad \text{---} \; (39)$$

For operators $A : \mathcal{E}_1 \to \mathcal{E}_1$ and $B : \mathcal{E}_2 \to \mathcal{E}_2$

such that

$$A_{ij} = \langle a_i | A | a_j \rangle \equiv \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \quad (40)$$

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$$B_{kl} = \langle b_k | B | b_l \rangle \equiv \begin{pmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & & \\ B_{m1} & \cdots & B_{mm} \end{pmatrix} \quad (41)$$

then

$$C = A \otimes B$$

or

$$C_{ik;jl} = \langle a_i b_k | A \otimes B | a_j b_l \rangle$$

$$= \langle a_i | A | a_j \rangle \langle b_k | B | b_l \rangle$$

or

$$C_{ik;jl} = A_{ij} B_{kl} \quad (42)$$

If the ordered basis for $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$ is $\{ |a_1 b_1\rangle \cdots |a_1 b_m\rangle, |a_2 b_1\rangle \cdots |a_2 b_m\rangle \cdots |a_n b_m\rangle \}$

then

$$C = A \otimes B = \begin{pmatrix} a_{11} B & \cdots & a_{1n} B \\ \vdots & & \vdots \\ a_{n1} B & & a_{nn} B \end{pmatrix} \quad (43)$$

where

$$a_{ij} B = \begin{pmatrix} a_{ij} b_{11} & \cdots & a_{ij} b_{1m} \\ \vdots & & \\ a_{ij} b_{m1} & \cdots & a_{ij} b_{m} \end{pmatrix} \quad (44)$$

For example, if

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

then

$$\cancel{A} \; C = A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

Addition of Angular Momenta:

Suppose we have two distinct angular momenta J_1 and J_2 which may ~~correspond~~ belong to different particles of a given system or may correspond to two different types of angular momenta (say L and S) of the same particle. ~~If such a system is rotated about~~ If the coordinate system describing such a system is rotated by an angle ϕ about an axis $\hat{n}$, the rotation operator for the composite system will be a direct product of the individual rotation operators

$$U_R^{\varepsilon} = U_R^{(\varepsilon_1)} \otimes U_R^{(\varepsilon_2)} \quad (45)$$

where ε_1 is the state space of J_1 and ε_2 that of J_2, while

$$\varepsilon = \varepsilon_1 \otimes \varepsilon_2$$

Thus

$$U_R^{(\varepsilon)} = e^{-\frac{i}{\hbar}\vec{J}_1 \cdot \hat{n}\phi} \otimes e^{-\frac{i}{\hbar}\vec{J}_2 \cdot \hat{n}\phi} \quad (46)$$

$$U_R^{(\varepsilon)} = e^{-\frac{i}{\hbar}\hat{n}\cdot(\vec{J}_1 \otimes 1 + 1 \otimes \vec{J}_2)\phi} \quad (47)$$

~~thus~~ $U_R^{(\varepsilon_1)}$ and $U_R^{(\varepsilon_2)}$ operate in state spaces ε_1 and ε_2, respectively, while $U_R^{(\varepsilon)}$ operates in the direct product space $\varepsilon = \varepsilon_1 \otimes \varepsilon_2$. similarly total angular momentum operator

$$\vec{J} = \vec{J}_1 \otimes 1 + 1 \otimes \vec{J}_2 \quad (48)$$

While Eq. (48) is the correct definition of the total angular momentum operator, ~~while~~ often it is written as

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \qquad \text{———} \quad (49)$$

~~Ex~~ Given the fact that $[\vec{J}_1, \vec{J}_2] = 0$, it is easy to verify J^2 and the components $\vec{J}$ will satisfy the commutation relations satisfied by $\vec{J}_1$ and $\vec{J}_2$

$$[J^2, J_i] = 0 \qquad \text{———} \quad (50)$$

$$[J_i, J_j] = i\hbar\, \varepsilon_{ijk} J_k \qquad \text{———} \quad (51)$$

Thus there must exist a basis $|jm\rangle$ which ~~is made up~~ forms common eigenvectors of J^2 and J_z

$$J^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle \quad \text{—} \quad (52)$$

$$J_z |jm\rangle = m\hbar |jm\rangle \qquad \text{———} \quad (53)$$

but it is easy to verify that

$$[J_z, J_1^2] = [J_z, J_2^2] = [J^2, J_1^2] = [J^2, J_2^2] = 0 \quad (54)$$

This means that $|jm\rangle$ should also be eigenvectors of J_1^2 and J_2^2, in addition

J^2 and J_z ;

$$J_1^2 |jm\rangle = j_1(j_1+1)\hbar^2 |jm\rangle$$
$$J_2^2 |jm\rangle = j_2(j_2+1)\hbar^2 |jm\rangle \quad \Bigg\} - (55)$$

therefore we adopt the notation

$$|jm\rangle \longrightarrow |j_1 j_2 j m\rangle \quad -(56)$$

which indicates that it is an eigenvector of J_1^2, J_2^2, J^2, and J_z. Because $\vec{J}$ is an operator in the direct product space $\mathcal{E}_1 \otimes \mathcal{E}_2$, we should be able to express $|j_1 j_2 j m\rangle$ as a linear combination of the basis

$$|j_1 j_2 m_1 m_2\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

$$|j_1 j_2 j m\rangle = \mathbb{1} |j_1 j_2 j m\rangle$$

or

$$|j_1 j_2 j m\rangle = \sum_{m_1, m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle \quad -(57)$$

coefficients in this linear expansion $\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$ are called Clebsch-Gordon coefficients.

Let us apply $J_z = J_{z1} + J_{z2}$ on both the sides of Eq. (57)

$$\cancel{J_z |j_1 j_2 J m\rangle = (J_{1z} + J_{2z}) \sum}$$

$$J_z |j_1 j_2 J m\rangle = \sum_{m_1, m_2} (J_{1z} + J_{2z}) |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 J m\rangle$$

$$m\hbar |j_1 j_2 J m\rangle = \sum_{m_1, m_2} (m_1 + m_2)\hbar |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 J m\rangle$$

On taking the inner product of this equation with the ket $|j_1 j_2 m_1' m_2'\rangle$, we have

$$m \langle j_1 j_2 m_1' m_2' | j_1 j_2 J m\rangle = (m_1' + m_2') \langle j_1 j_2 m_1' m_2' | j_1 j_2 J m\rangle$$

or

$$m \langle j_1 j_2 m_1 m_2 | j_1 j_2 J m\rangle = (m_1 + m_2) \langle j_1 j_2 m_1 m_2 | j_1 j_2 J m\rangle$$

$$\Rightarrow \boxed{\left.\begin{array}{l} \langle j_1 j_2 m_1 m_2 | j_1 j_2 J m\rangle \neq 0 \\ \quad \text{only if} \quad m = m_1 + m_2 \end{array}\right\}} \quad -(58)$$

By applying J_+ / J_- to both sides of Eq. (57), we can derive important recursion relations satisfied by CG coefficients

$$J_\pm |j_1 j_2 j m\rangle = \sum_{m_1, m_2} (J_{\pm 1} + J_{\pm 2}) |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

but

$$J_\pm |j_1 j_2 j m\rangle = \sqrt{(j \mp m)(j \pm m+1)}\,\hbar\, |j_1 j_2 j\, m\pm 1\rangle$$

and

$$(J_{\pm 1} + J_{\pm 2}) |j_1 j_2 m_1 m_2\rangle$$
$$= \sqrt{(j_1 \mp m_1)(j_1 \pm m_1+1)}\,\hbar\, |j_1 j_2\, m_1 \pm 1, m_2\rangle$$
$$+ \sqrt{(j_2 \mp m_2)(j_2 \pm m_2+1)}\,\hbar\, |j_1 j_2\, m_1\, m_2 \pm 1\rangle$$

using these we obtain

$$\sqrt{(j \mp m)(j \pm m+1)}\, |j_1 j_2 j\, m\pm 1\rangle$$
$$= \sum_{m_1, m_2} \sqrt{(j_1 \mp m_1)(j_1 \pm m_1+1)}\, |j_1 j_2\, m_1 \pm 1, m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$
$$+ \sum_{m_1, m_2} \sqrt{(j_2 \mp m_2)(j_2 \pm m_2+1)}\, |j_1 j_2\, m_1\, m_2 \pm 1\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle \quad \text{--- (59)}$$

On changing the summed m values m_1 and m_2 to m_1' and m_2' and taking the inner product of Eq. (59) with $|J_1 J_2 m_1 m_2\rangle$, we obtain

$$\sqrt{(J\mp m)(J\mp m+1)}\,\langle J_1 J_2 m_1 m_2 | J_1 J_2 J\, m\mp 1\rangle$$

$$= \sum_{m_1' m_2'} \sqrt{(J_1\mp m_1')(J_1\mp m_1'+1)}\; \delta_{m_1, m_1'\pm 1}\, \delta_{m_2 m_2'} \langle J_1 J_2 m_1' m_2' | J_1 J_2 J m\rangle$$

$$+ \sum_{m_1' m_2'} \sqrt{(J_2\mp m_2')(J_2\pm m_2'+1)}\; \delta_{m_1' m_1}\, \delta_{m_2', m_2'\pm 1} \langle J_1 J_2 m_1' m_2' | J_1 J_2 J m\rangle$$

$$= \sqrt{(J_1\mp(m_1\mp 1))(J_1\pm(m_1\mp 1)+1)}\,\langle J_1 J_2\, m_1\mp 1\, m_2 | J_1 J_2 J m\rangle$$

$$+ \sqrt{(J_2\mp(m_2\mp 1))(J_2\pm(m_2\mp 1)+1)}\,\langle J_1 J_2\, m_1\, m_2\mp 1 | J_1 J_2 J m\rangle$$

~~0ro~~ Finally we obtain the important recursion relation involving C-G coefficients

$$\sqrt{(J\mp m)(J\pm m+1)}\,\langle J_1 J_2 m_1 m_2 | J_1 J_2\, m\pm 1\rangle$$

$$= \sqrt{(J_1\mp m_1+1)(J_1\pm m_1)}\,\langle J_1 J_2\, m_1\mp 1\, m_2 | J_1 J_2 J m\rangle$$

$$+ \sqrt{(J_2}$$

$$\boxed{\begin{aligned}&\sqrt{(j \mp m)(j \pm m+1)}\ \langle j_1\, j_2\, m_1\, m_2 | j_1\, j_2\, j\, m \pm 1\rangle \\ &= \sqrt{(j_1 \mp m_1+1)(j_1 \pm m_1)}\ \langle j_1\, j_2\, m_1 \mp 1\, m_2 | j_1\, j_2\, j\, m\rangle \\ &+ \sqrt{(j_2 \mp m_2+1)(j_2 \pm m_2)}\ \langle j_1\, j_2\, m_1\, m_2 \mp 1 | j_1\, j_2\, j\, m\rangle\end{aligned}} \qquad (60)$$

Using Recursion Relations to compute C-G Coefficients:

Using recursion relations (Eq. 60), one can, for a given set of j_1, j_2, and j, all non vanishing C-G coefficients just in terms of one of them.

Let us choose $m_1 = j_1$, $m = j$, and $m_2 = j - j_1 - 1$, ~~$m_2 = j - j_1$~~ and use the lower sign in (60)

$$\sqrt{2j(j-j+1)}\ \langle j_1\, j_2\, j_1,\, j-j_1-1 | j_1\, j_2\, j\, j-1\rangle$$

$$= \sqrt{(2j_1+1)(j_1-j_1)}\ \langle j_1\, j_2\, j_1+1,\, j-j_1-1$$

$$\sqrt{(2J)(J-J+1)}\ \langle j_1 j_2\ j_1\ J-j_1-1 | j_1 j_2\ J\ J-1\rangle$$

$$= \sqrt{(2j_1+1)(j_1-j_1)}\ \langle j_1 j_2\ j_1+1\ J-j_1-1 | j_1 j_2\ J\ J\rangle$$

$$+ \sqrt{(j_2+J-j_1-1+1)(j_2-J+j_1-1)}\ \langle j_1 j_2\ j_1\ J-j_1 | j_1 j_2\ J\ J\rangle$$

$$\Rightarrow \boxed{\langle j_1 j_2\ j_1\ J-j_1-1 | j_1 j_2\ J\ J-1\rangle = \sqrt{\frac{(j_2+J-j_1)(j_2-J+j_1-1)}{2J}}\ \langle j_1 j_2\ j_1\ J-j_1 | j_1 j_2\ J\ J\rangle} \quad (61)$$

Thus, using (61) one can compute $\langle j_1 j_2\ j_1\ J-j_1-1 | j_1 j_2\ J\ J-1\rangle$ provided $\langle j_1 j_2\ j_1\ J-j_1 | j_1 j_2\ J\ J\rangle$ is known.

Let us use the upper ~~sign~~ sign in Eq. (60) and $m_1 = j_1$, $m_2 = J-j_1$, $m = J-1$

$$\sqrt{(1)(J+J-1+1)}\ \langle j_1 j_2\ j_1\ J-j_1 | j_1 j_2\ J\ J\rangle$$

$$= \sqrt{(j_1-j_1+1)(j_1+j_1)}\ \langle j_1 j_2\ j_1-1\ J-j_1 | j_1 j_2\ J\ J-1\rangle$$

$$+ \sqrt{(j_2-J+j_1+1)(j_2+J-j_1)}\ \langle j_1 j_2\ j_1\ J-j_1-1 | j_1 j_2\ J\ J-1\rangle$$

$\Rightarrow$

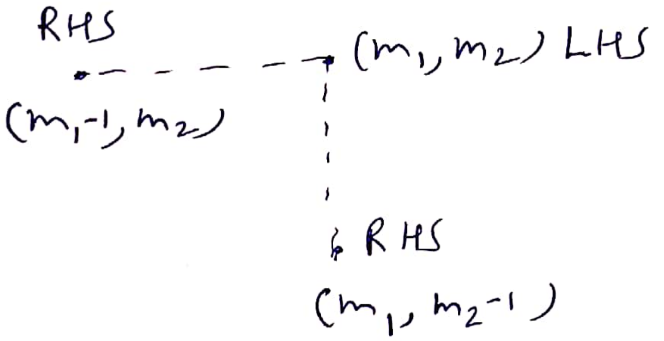
$$\boxed{\begin{aligned}&\langle j_1\, j_2\, j_1{-}1\;\; j-j_1 \mid j_1\, j_2\, j\; j-1\rangle \\ &= \sqrt{\frac{j}{j_1}}\;\langle j_1\, j_2\, j_1\;\; j-j_1 \mid j_1\, j_2\, j\; j\rangle \\ &\quad - \sqrt{\frac{(j_1+j_2-j)(j_1+j_2-j+1)}{2j_1}}\;\langle j_1\, j_2\, j_1\;\; j-j_1-1 \mid j_1\, j_2\, j\; j-1\rangle\end{aligned}}$$

(62)

Eq. (62) allows us to compute $\langle j_1 j_2\, j_1{-}1\;\; j-j_1 \mid j_1\, j_2\, j\; j-1\rangle$ using the values of $\langle j_1 j_2\, j_1\;\; j-j_1 \mid j_1\, j_2\, j\, j\rangle$ and $\langle j_1 j_2\, j_1\;\; j-j_1-1 \mid j_1 j_2\, j\; j-1\rangle$. Thus, using Eq. (60), we can compute all the C-G coefficients provided we know ~~that~~ the value of $\langle j_1\, j_2\, j_1\;\; j-j_1 \mid j_1 j_2\, j\, j\rangle$.

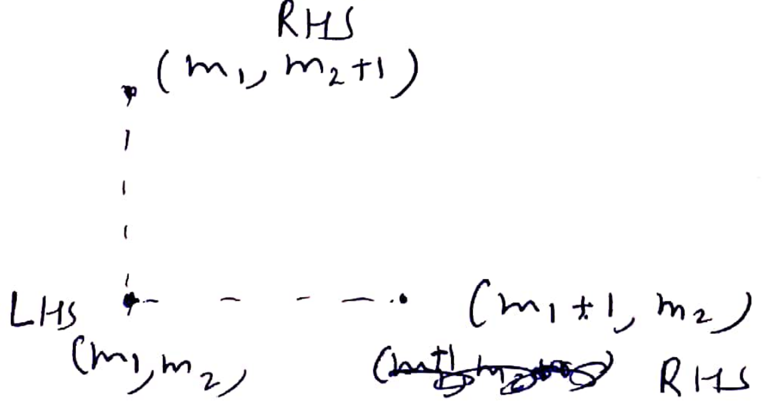
We can give a geometric interpretation to the recursion relations of Eq. (60). Let us assume that there is a discrete plane on which quantum numbers m_1 and m_2 exist, and a point on the plane is denoted by ~~(m~~ the ordered pair (m_1, m_2).

If we start with a C-G coefficient with starting values (m_1, m_2) then the upper sign connects it with (m_1-1, m_2) and (m_1, m_2-1) on the RHS. This is shown pictorially as



~~likewise likewise the lower sign~~

likewise the lower sign corresponds to



Triangular Inequality:

Let us derive another important result which allows us to compute allowed values of J, for a given set of values of J_1 and J_2. Let us consider the C-G coefficient $\langle J_1 J_2 J_1\ J-J_1 | J_1 J_2 J J \rangle$. Because $J-J_1$ is a possible value of m_2 so it ~~mas~~ must satisfy

$$-J_2 \leq J - J_1 \leq J_2$$

$$\Rightarrow \qquad J_1 - J_2 \leq J \leq J_1 + J_2 \qquad \text{— (63a)}$$

~~But so~~ If in the considered C-G coefficient we interchange $J_1 \leftrightarrow J_2$ so that $\langle J_1 J_2 | J_1\ J-J_1 | J_1 J_2\ J J \rangle$

$$\longrightarrow \langle J_2 J_1\ J_2\ J-J_1 | J_2 J_1\ J J \rangle$$

If instead we consider the C-G coefficient $\langle J_1 J_2\ J-J_2\ J_2 | J_1 J_2\ J J \rangle$, then for it to have nonzero values

$$-J_1 \leq J - J_2 \leq J_1$$

$$\Rightarrow \qquad J_2 - J_1 \leq J \leq J_1 + J_2 \qquad \text{— (63b)}$$

Noting the fact that j cannot take negative values, (63a) and (63b) can be combined into a single inequality

$$\boxed{|j_1 - j_2| \leq j \leq j_1 + j_2} \quad \text{—(64)}$$

which is called the triangular inequality.

Orthonormality Conditions:

Next we derive two orthonormality conditions involving C-G coefficients. We know

$$\langle j_1 j_2 j m | j_1 j_2 j' m' \rangle = \delta_{jj'} \delta_{mm'}$$

$$\Rightarrow \quad \sum_{m_1, m_2} \langle j_1 j_2 j m | j_1 j_2 m_1 m_2 \rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j' m' \rangle = \delta_{jj'} \delta_{mm'}$$

assuming that all C-G coefficients are real so that

$$\langle j_1 j_2 j m | j_1 j_2 m_1 m_2 \rangle = \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle$$

so that

$$\boxed{\sum_{m_1, m_2} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j' m' \rangle = \delta_{jj'} \delta_{mm'}} \quad \text{—(65)}$$

the second orthonormality condition is obtained from

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 m_1' m_2' \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

$$\Rightarrow \sum_{jm} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle \langle j_1 j_2 j m | j_1 j_2 m_1' m_2' \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

$$\Rightarrow \boxed{\sum_{jm} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}} \qquad (66)$$

which is the other condition.

From (65) and (66) it is obvious that C-G coefficients form a unitary matrix.

For Normalization conditions obtainable from Eqs. (65) and (66), recursion relations are used to compute the C-G-C $\langle j_1 j_2 j_1\, j-j_1 | j_1 j_2 j j \rangle$, which by convention is taken to be real and positive. Rest of the C-G-coefficients (CGCs) are generated from that using the recursion relations.

Examples: (i) Compute CGC $\langle j 1 j 0 | j 1 j j \rangle$

Soln: In this CGC $j_1 = j$, $j_2 = 1$, $m_1 = j$, $m_2 = 0$, $j = j$ and $m = j$. Clearly it is of the form $\langle j_1 j_2 j_1\, j - j_1 | j_1 j_2\, j j\rangle$. Thus we ~~from the~~ should use normalization Eq. (65) with $j = j'$ and $m = m'$ leading to

$$\sum_{m_1 m_2} \langle j 1 m_1 m_2 | j 1 j j\rangle^2 = 1$$

$$\Rightarrow \sum_{m_1} \langle j 1 m_1 1 | j 1 j j\rangle^2 + \sum_{m_1} \langle j 1 m_1 0 | j 1 j j\rangle^2 + \sum_{m_1} \langle j 1 m_1 -1 | j 1 j j\rangle^2 = 1$$

$$\because \quad m_1 + m_2 = j$$

$$\Rightarrow \langle j 1\, j-1\, 1 | j 1 j j\rangle^2 + \langle j 1 j 0 | j 1 j j\rangle^2 = 1 \qquad (67)$$

In recursion relation (60) with upper signs and $m = j$, $j_1 = j$, $j_2 = 1$ $m_1 = j$, $m_2 = 1$ we obtain

$$\sqrt{2j}\, \langle j 1\, j-1\, 1 | j 1 j j\rangle = -\sqrt{2}\, \langle j 1 j 0 | j 1 j j\rangle$$

using this in Eq. (67), we obtain

$$\left\{1+\frac{1}{J}\right\}\langle J1J0|J1JJ\rangle^2 = 1$$

$$\Rightarrow \boxed{\langle J1J0|J1JJ\rangle = \sqrt{\frac{J}{J+1}}}$$

by convention we have chosen the CGC above to be real and positive.

(ii) Calculate $\langle J2J0|J2JJ\rangle$

Soln: This CGC is also of the form $\langle J_1 J_2 J_1, J-J_1 | J_1 J_2 J J\rangle$ with $J_1 = J$, $J_2 = 2$, $J = J$, $m_1 = J$ and $m = J$. Using the normalization condition ~~from~~ of Eq. (65), we obtain

$$\sum_{m_1 m_2} |\langle J2 m_1 m_2 | J2JJ\rangle|^2 = 1$$

m_2 has possible values 0, ±1, ±2.

~~⇒ 0 |⟨J2J0|J2JJ⟩|² + |⟨J2~~

Because $m_1 + m_2 = J$ so negative values of m_2 will require $m_1 > J = J_1$ which is not possible. Hence only allowed terms in the sum above are

$$\langle J\,2\,J\,0|J\,2\,J\,J\rangle^2 + \langle J\,2\,J\!-\!1\,1|J\,2\,J\,J\rangle^2 + \langle J\,2\,J\!-\!2\,2|J\,2\,J\,J\rangle^2 = 1$$

using the upper sign in Eqn. (60) and

$j_1 = j = J$, $m = J$, $j_2 = 2$, $m_1 = J$, $m_2 = 1$,

we obtain

$$\sqrt{(J+J)(J-J+1)}\,\langle J\,2\,J\!-\!1\,1|J\,2\,J\,J\rangle + \sqrt{(2+1)(2-1+1)}\,\langle J\,2\,J\,0|J\,2\,J\,J\rangle = 0$$

$$\Rightarrow \quad \langle J\,2\,J\!-\!1\,1|J\,2\,J\,J\rangle = -\sqrt{\frac{3}{J}}\,\langle J\,2\,J\,0|J\,2\,J\,J\rangle$$

So in the same eqn if we use

$j_1 = j = J$, $m = J$, $j_2 = 2$, $m_1 = J-1$, $m_2 = 2$,

obtain

$$\sqrt{(J+J-1)(J-J+1+1)}\,\langle J\,2\,J\!-\!2\,2|J\,2\,J\,J\rangle + \sqrt{(2+2)(2-2+1)}\,\langle J\,2\,J\!-\!1\,1|J\,2\,J\,J\rangle$$

$$\Rightarrow \quad \boxed{\langle J\,2\,J\!-\!2\,2|J\,2\,J\,J\rangle = -\sqrt{\frac{2}{(2J-1)}}\,\langle J\,2\,J\!-\!1\,1|J\,2\,J\,J\rangle}$$

$$\Rightarrow \quad \langle J\,2\,J\!-\!2\,2|J\,2\,J\,J\rangle = \frac{\sqrt{3}}{J}\,\langle J\,2\,J\,0|J\,2\,J\,J\rangle$$

$$\Rightarrow \langle j\,2\,j-2\,2 \mid j\,2\,j\,j\rangle = -\sqrt{\frac{2}{2j-1}}\,\langle j\,2\,j-1\,1 \mid j\,2\,j\,j\rangle$$

$$\Rightarrow \langle j\,2\,j-2\,2 \mid j\,2\,j\,j\rangle = \sqrt{\frac{6}{j(2j-1)}}\,\langle j\,2\,j\,0 \mid j\,2\,j\,j\rangle$$

using these in the normalization sum we obtain

$$\langle j\,2\,j\,0 \mid j\,2\,j\,j\rangle^2 \left\{ 1 + \frac{3}{j} + \frac{6}{j(2j-1)} \right\} = 1$$

$$\langle j\,2\,j\,0 \mid j\,2\,j\,j\rangle^2 \left\{ \frac{2j^2+5j+3}{j(2j-1)} \right\} = 1$$

$$\langle j\,2\,j\,0 \mid j\,2\,j\,j\rangle^2 \left\{ \frac{(2j+3)(j+1)}{j(2j-1)} \right\} = 1$$

$$\Rightarrow \boxed{\langle j\,2\,j\,0 \mid j\,2\,j\,j\rangle = \sqrt{\frac{j(2j-1)}{(2j+3)(j+1)}}}$$

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The Clebsch-Gordon Series:

In the previous discussion about the coupling of angular angular momenta, we showed that the direct product basis states $|j_1 j_2 m_1 m_2\rangle$ are identical in number, and equivalent to the coupled basis states $|j_1 j_2 j m\rangle$, with the ~~Clebsch~~ CGCs forming the unitary matrix connecting the two basis sets. ~~Another way to~~ There are several consequences of this. For example, the resolution of identity in the two basis is related as

$$\sum_{m_1, m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2|$$

$$= \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} |j_1 j_2 j m\rangle\langle j_1 j_2 j m|$$

$$= \sum_m \Big\{ |j_1 j_2 \, |j_1-j_2| \, m\rangle\langle j_1 j_2 \, |j_1-j_2| \, m| \oplus |j_1 j_2 \, |j_1-j_2|+1 \, m\rangle\langle j_1 j_2 \, |j_1-j_2|+1 \, m| \oplus \cdots \oplus |j_1 j_2 \, j_1+j_2 \, m\rangle\langle j_1 j_2 \, j_1+j_2 \, m| \Big\} \tag{68}$$

The sum in the LHS is the direct sum, with the individual identities being in the block-diagonal form.

A similar result holds for the rotation matrices in the two basis

$$D^{(j_1)}(R) \otimes D^{(j_2)}(R) = D^{(|j_1-j_2|)}(R) \oplus D^{(|j_1-j_2|+1)}(R) \oplus \cdots \oplus D^{(j_1+j_2)}(R) \qquad (69)$$

which in the matrix form can be written as

$$D^{(j_1)} \otimes D^{(j_2)} \equiv \begin{pmatrix} D^{(|j_1-j_2|)} & & & 0 \\ & D^{(|j_1-j_2|+1)} & & \\ & & \ddots & \\ 0 & & & D^{(j_1+j_2)} \end{pmatrix} \qquad (70)$$

The exact quantitative relationship denoting Eqs. (69) and (70) is called the Clebsch-Gordon series, written as

$$D^{(j_1)}_{m_1' m_1}(R)\, D^{(j_2)}_{m_2' m_2}(R) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m, m'} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m' \rangle D^{(j)}_{m' m}(R) \qquad (71)$$

<u>Proof:</u> It is obvious

$$\cancel{\langle j_1 j_2 m_1' m_2' | D(R) | j_1 j_2 m_1 m_2 \rangle = \langle j_1 m_1' | D(R) | j_1 m_1 \rangle \times}$$

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Proof: It is obvious that

$$\langle j_1 j_2 m_1' m_2' | U_R | j_1 j_2 m_1 m_2 \rangle$$

$$= \langle j_1 m_1' | U_R | j_1 m_1 \rangle \langle j_2 m_2' | U_R | j_2 m_2 \rangle$$

$$= D^{(j_1)}_{m_1' m_1}(R)\, D^{(j_2)}_{m_2' m_2}(R) \qquad \text{——} \quad (72)$$

Inserting resolution of identity two times on the ~~left~~ LHS of Eq. (72), we have

$$D^{(j_1)}_{m_1' m_1}(R)\, D^{(j_2)}_{m_2' m_2}(R) = \sum_{jm}\sum_{j'm'} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j' m' \rangle$$

$$\langle j_1 j_2 j' m' | U_R | j_1 j_2 j m \rangle$$

$$\langle j_1 j_2 j m | j_1 j_2 m_1 m_2 \rangle$$

but

$$\langle j_1 j_2 j' m' | U_R | j_1 j_2 j m \rangle = \delta_{jj'} D^{(j)}_{m' m}(R)$$

substituting it above, we have

$$D^{(j_1)}_{m_1' m_1}(R)\, D^{(j_2)}_{m_2' m_2} = \sum_{j}\sum_{m' m} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m' \rangle$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle$$

$$D^{(j)}_{m' m}(R).$$

QED

Tensor Operators:

We will first discuss vector operators, and then generalize the discussion to define ~~the~~ tensor operators.

Let us assume that there is an operator $\bar{A}$, called a vector operator, because its expectation value rotates as ~~though~~ per rules of transformation of a vector, under an active rotation R. If the system is in state $|\psi\rangle$, we know that under the rotation it will transform into $|\psi'\rangle$ as

$$|\psi'\rangle = U_R|\psi\rangle .$$

If $\hat{e}$ is an arbitrary unit vector then clearly $\bar{A}\cdot\hat{e}$ is a scalar and hence its value will be invariant under the rotation

$$\langle\psi|\bar{A}\cdot\hat{e}|\psi\rangle = \langle\psi'|\bar{A}\cdot\hat{e}'|\psi'\rangle$$

$$\Rightarrow \cancel{\langle\psi|\bar{A}\cdot\hat{e}|\psi\rangle = \langle\psi|U_R^{\dagger}\bar{A}\cdot\hat{e}'U_R|\psi\rangle}$$

$$\langle\psi'|U_R\,\bar{A}\cdot\hat{e}\,U_R^{\dagger}|\psi'\rangle = \langle\psi'|\bar{A}\cdot\hat{e}'|\psi'\rangle$$

$$\Rightarrow \quad U_R \bar{A} U_R^{\dagger} \cdot \hat{e} = \bar{A} \cdot \hat{e}' \qquad (73)$$

~~above $\hat{e}'$ denote~~ $\hat{e}$ transforms in $\hat{e}'$ after the rotation. But, in a Cartesian basis

$$\hat{e}'_i = \sum_j R_{ij}\, e_j \qquad (74)$$

upon substituting this above, we obtain

$$\sum_k U_R A_k U_R^{\dagger} \hat{e}_k = \sum_i A_i \hat{e}'_i$$

$$\sum_k U_R A_k U_R^{\dagger} \hat{e}_k = \sum_{ij} A_i R_{ij} \hat{e}_j$$

comparing coefficients of $\hat{e}_j$ on both the sides, we have

$$\boxed{U_R A_j U_R^{\dagger} = \sum_i A_i R_{ij}} \qquad (75)$$

Let us consider an infinitesimal rotation about an axis $\hat{n}$, by an angle ϵ. Then

$$U_R = e^{-\frac{i(\bar{J}\cdot\hat{n})\epsilon}{\hbar}} \approx 1 - \frac{i(\bar{J}\cdot\hat{n})\epsilon}{\hbar} \qquad (76)$$

and we know that under such a rotation

$$\hat{e} \rightarrow \hat{e}' \simeq \hat{e} + \epsilon\, \hat{n} \times \hat{e} \quad — (77)$$

Substituting (77) ~~ne~~ and (76) in (73)

$$\left(1 - i\frac{\vec{J}\cdot\hat{n}}{\hbar}\epsilon\right) \vec{A}\cdot\hat{e} \left(1 + \frac{i\vec{J}\cdot\hat{n}}{\hbar}\epsilon\right)$$

$$= \vec{A}\cdot\hat{e} + \epsilon\; \vec{A}\cdot(\hat{n}\times\hat{e})$$

$$= \vec{A}\cdot\hat{e} + \epsilon\; (\vec{A}\times\hat{n})\cdot\hat{e}$$

$$= (\vec{A} - \epsilon\, \hat{n}\times\vec{A})\cdot\hat{e}$$

Neglecting terms $O(\epsilon^2)$, and comparing other terms on both the sides, we obtai

$$-\frac{i}{\hbar}\epsilon\, \vec{J}\cdot\hat{n}\,\vec{A} + \frac{i}{\hbar}\epsilon\, \vec{A}\, \vec{J}\cdot\hat{n} = -\epsilon\, \hat{n}\times\vec{A}$$

$$\boxed{[\vec{A}, \vec{J}\cdot\hat{n}] = i\hbar\, \hat{n}\times\vec{A}} \quad — (78)$$

using Einstein convention, we have

$$\vec{J}\cdot\hat{n} = J_j n_j$$

$$(\hat{n}\times A)_i = \varepsilon_{ijk}\, n_j A_k$$

on substituting these above for the i-th component of Eq. (78), we obtain

$$\boxed{[A_i, J_j] = i\hbar\, \varepsilon_{ijk} A_k} \quad \text{—(79)}$$

Next we define a tensor operator as a generalization of a vector operator.

~~∅~~ We saw that a vector operator $\bar{A}$ transforms according to Eq. (75) under a rotation. We define a spherical tensor operator T_k^q, with $q = -k, -k+1, \ldots k-1, k$, of rank k as the operator which transforms according ~~to~~ to the rule

$$\boxed{U_R T_k^q U_R^\dagger = \sum_{q'=-k}^{k} T_k^{q'} D^{(k)}_{q'q}(R)} \quad \text{—(80)}$$

According to this definition, an object of rank $k=1$, is a spherical vector.

Let us see ~~whether~~ how $Y_\ell^m(\theta, \phi)$ transform ~~[illegible]~~ for $\ell = 1$. We saw earlier

$$Y_\ell^m(\theta', \phi') = \sum_{m'=-\ell}^{\ell} Y_\ell^{m'}(\theta, \phi) D^{(\ell)}_{m'm} \quad \text{—(81)}$$

where $Y_\ell^m(\theta', \phi')$ is the same function with respect to the rotated coordinate system. Noting that

$$U_R Y_\ell^m(\theta,\phi) U_R^\dagger = Y_\ell^m(\theta',\phi')$$

we find that Eq. (81) and (80) have the same from. Thus Y_ℓ^m's are tensor operators of rank ℓ. Coming back to the case of $\ell = 1$, we have

$$Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\, e^{\pm i\phi} \sin\theta$$

$$= \mp\sqrt{\frac{3}{8\pi}}\left(\sin\theta\cos\phi \pm i \sin\theta \sin\phi\right)$$

$$\boxed{Y_1^{\pm}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\,\frac{(x \pm iy)}{r}}$$

and

$$\boxed{Y_1^{0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}} \quad (82)$$

So for this case Eq. (81) yields

$$\boxed{\left(-\frac{x'+iy'}{\sqrt{2}},\, z',\, \frac{x'-iy'}{\sqrt{2}}\right) = \left(-\frac{x+iy}{\sqrt{2}},\, z,\, \frac{x-iy}{\sqrt{2}}\right) D^{(1)}(R)} \quad (83)$$

using this, we can define the components of a spherical tensor T_1^q, when the Cartesian components of the vector (A_x, A_y, A_z) are given

$$\boxed{\begin{aligned} T_1^{\pm} &= \mp \frac{A_x \pm i A_y}{\sqrt{2}} \\ T_1^{0} &= A_z \end{aligned}} \quad \text{--- (84)}$$

Commutation Relations:

For an infinitesimal rotation of angle ϵ, about an axis along the direction $\hat{n}$, we have from Eq. (80)

$$\left(1 - \frac{i}{\hbar} \vec{J}\cdot\hat{n}\,\epsilon\right) T_k^q \left(1 + \frac{i}{\hbar} \vec{J}\cdot\hat{n}\,\epsilon\right)$$

$$= \sum_{q'=-k}^{k} T_k^{q'} \langle k q' | 1 - \frac{i}{\hbar} \vec{J}\cdot\hat{n}\,\epsilon | k q \rangle$$

above we used the fact that $D^{(k)}_{q'q}(R) = \langle k q' | U_R | k q \rangle$, and for an infinitesimal rotation $U_R \approx 1 - \frac{i}{\hbar} \vec{J}\cdot\hat{n}\,\epsilon$.

$$\Rightarrow \quad T_k^q + \frac{i}{\hbar} [T_k^q, \vec{J}\cdot\hat{n}]\epsilon + O(\epsilon^2) = T_k^q - \frac{i}{\hbar}\epsilon \sum_{q'=-k} T_k^{q'} \langle k q' | \vec{J}\cdot\hat{n} | k q \rangle$$

$$\Rightarrow \boxed{[\hat{n}\cdot\bar{J}, T_k^q] = \sum_{q'=-k}^{k} T_k^{q'} \langle kq'|\hat{n}\cdot\bar{J}|kq\rangle} \quad (85)$$

taking $\hat{n} = \hat{k}$, we obtain above

$$\boxed{[J_z, T_k^q] = q\hbar\, T_k^q} \quad (86)$$

and $\hat{n} = \hat{n}_\pm = \hat{i} \pm i\hat{j}$. so that

$$\hat{J}\cdot n = J_\pm$$

and using the fact that

$$J_\pm |kq\rangle = \sqrt{(k\mp q)(k\pm q+1)}\, |k\, q\pm 1\rangle$$

we obtain

$$\boxed{[J_\pm, T_k^q] = \hbar\sqrt{(k\mp q)(k\pm q+1)}\; T_k^{q\pm 1}} \quad (87)$$

Eqs. (86) & (87) are fundamental commutation relations of tensor operators

Wigner - Eckart Theorem :

Let us compute the matrix elements of both sides of Eq. (80), with respect to angular momentum eigenstates $|\alpha j m\rangle$ and $|\alpha' j' m'\rangle$, where α, α' quantum numbers other than angular momentum which are needed to specify these states completely

$$\langle \alpha' j' m'| U_R T_k^q U_R^\dagger |\alpha j m\rangle = \sum_{q'=-k}^{k} \langle \alpha' j' m'| T_k^{q'} |\alpha j m\rangle D^{(k)}_{q'q}$$

Using resolution of identity two times on the LHS, we have

$$\cancel{\sum_{m'}\sum_{m} \langle \alpha' j' m' | \alpha' j' m'\rangle \langle \alpha' j' m'|}$$

$$\sum_{m, m'} \langle \alpha' j' m' | U_R | \alpha' j' m' \rangle \langle \alpha' j' m' | T_k^q | \alpha j m \rangle \langle \alpha j m | U_R^\dagger | \alpha j m \rangle$$

$$= \sum_{q'=-k}^{k} \langle \alpha' j' m' | T_k^{q'} | \alpha j m \rangle D^{(k)}_{q' q}$$

but

$$\langle \alpha' j' m' | U_R | \alpha' j' m' \rangle = D^{(j')}_{m' m'}(R)$$

$$\langle \alpha j m | U_R^\dagger | \alpha j m \rangle = D^{(j)*}_{m m}(R)$$

we obtain

$$\boxed{\sum_{m m'} D^{(j')}_{m' m'} \langle \alpha' j' m' | T_k^q | \alpha j m \rangle D^{(j)*}_{m m}(R) = \sum_{q'=-k}^{k} \langle \alpha' j' m' | T_k^{q'} | \alpha j m \rangle D^{(k)}_{q' q}(R)} \quad - (88)$$

we can recast C-G series of Eq.(71) as

$$\boxed{\begin{aligned}&\sum_{\mu\mu'} D^{(j')}_{m'\mu'} \langle jk\,\mu q | jk\, j'\mu'\rangle D^{(j)\,*}_{m\mu}(R) \\ &\quad = \sum_{q'} \langle jk\, m q' | jk\, j' m'\rangle D^{(k)}_{q'q}(R)\end{aligned}} \qquad (89)$$

Eq.(88) can be seen as a linear homogeneous eqn for $\langle \alpha' j' m' | T_k^{q'} | \alpha j m\rangle$ and Eq (89) has the same coefficients except that it has unknowns $\langle jk\, m q' | jk\, j' m'\rangle$. Thus, the solutions of two equations must be proportional to each other. Thus

$$\boxed{\langle \alpha' j' m' | T_k^{q} | \alpha j m\rangle = \langle jk\, m q | jk\, j' m'\rangle \langle \alpha' j' \| T_k \| \alpha j\rangle} \qquad (90)$$

where the proportionality constant $\langle \alpha' j' \| T_k \| \alpha j \rangle$ is called the reduced matrix element and depends only on α, α', j, j', and k not on m, m', and q, because that dependence is contained in the C-G-C $\langle j k m q | j k j' m' \rangle$. Eq. (90) is called the Wigner-Eckart theorem.

The importance of Wigner-Eckart theorem lies in the fact that the required matrix element is written as a product of a CGC which contains the symmetry related information, and the reduced matrix element which contains information about other properties of the system.

Selection Rules : From the CGC involved in the Wigner-Eckart theorem (Eq. 90), we get two important selection rules which determine whether a given matrix element is zero, based just on the symmetry.

- We know that the CGC $\langle j k m q | j k j' m' \rangle$

is nonzero only if

(i) m selection rule is valid, i.e.,

$$m' = m + q$$

$$\Rightarrow \boxed{q = m' - m} \quad \text{--- (91)}$$

(ii) triangular identity is valid, i.e.,

$$|J - k| \leq J' \leq J + k$$

but triangular identity of numbers holds for all three numbers

$$\Rightarrow \boxed{|J - J'| \leq k \leq J + J'} \quad \text{--- (92)}$$

Examples: (i) for a scalar operator

$k = 0 \Rightarrow q = 0 \Rightarrow \Delta m = m' - m = 0$
and $\Delta J = J' - J = 0 \Rightarrow J = J'$

(ii) For a vector operator $k = 1$, $q = 0, \pm 1$

$\Rightarrow \Delta m = 0, \pm 1$ and $\Delta J = J' - J = 0, \pm 1$.

Projection Theorem: For a vector operator $\bar{A}$, with spherical components A_1^q or A^q for short,

$$\langle \alpha' j m' | A^q | \alpha j m \rangle = \frac{\langle \alpha' j m | \vec{J}\cdot\vec{A} | \alpha j m \rangle}{\hbar^2 j(j+1)} \times \langle j m' | J^q | j m \rangle$$

Proof: We have

$$\vec{J}\cdot\vec{A} = J_x A_x + J_y A_y + J_z A_z$$

$$= \frac{1}{2}(J_x + iJ_y)(A_x - iA_y) + \frac{1}{2}(J_x - iJ_y)(A_x + iA_y) + J_z A_z$$

$$= -J^{+1}A^{-1} - J^{-1}A^{+1} + J^0 A^0$$

where
$$A^{\pm 1} = \mp \frac{1}{\sqrt{2}}(A_x \pm iA_y)$$

$$A^0 = A_z$$

$$J^{\pm 1} = \mp \frac{1}{\sqrt{2}}(J_x \pm iJ_y) = \mp \frac{1}{\sqrt{2}} J_\pm$$

$$J^0 = J_z$$

with this

$$\langle \alpha' j m | \bar{J}.\bar{A} | \alpha j m \rangle = \langle \alpha' j m | J_0^0 A^0 - J^{+1} A^{-1} - J^{-1} A^{+1} | \alpha j m \rangle$$

$$= m\hbar \langle \alpha' j m | A^0 | \alpha j m \rangle$$

$$+ \frac{\hbar}{\sqrt{2}} \sqrt{(j+m)(j-m+1)} \langle \alpha' j m-1 | A^{-1} | \alpha j m \rangle$$

$$- \frac{\hbar}{\sqrt{2}} \sqrt{(j-m)(j+m+1)} \langle \alpha' j m+1 | A^{+1} | \alpha j m \rangle$$

But from Wigner-Eckart theorem

$$\langle \alpha' j m | A^0 | \alpha j m \rangle \propto \langle \alpha' j m | A^{+1} | \alpha j m \rangle$$

$$\propto \langle \alpha' j m | A^{-1} | \alpha j m \rangle \propto \langle \alpha' j || \bar{A} | \alpha j \rangle,$$

where $\langle \alpha' j || \bar{A} | \alpha j \rangle$ is the reduced matrix element of $\bar{A}$, independent of m and q.

Thus

$$\langle \alpha' j m | \bar{J}.\bar{A} | \alpha j m \rangle = C(j,m) \langle \alpha' j | \bar{A} | \alpha j \rangle$$

where $C(j,m)$ is a constant which depends on j and m, and is independent of α, α', and $\bar{A}$. ~~Fo~~ ~~Furthermore~~

Furthermore, $C(j,m)$ must be independent of m as well because $\vec{J}.\vec{A}$ is a scalar operator, thus

$$\langle \alpha' j m | \vec{J}.\vec{A} | \alpha j m \rangle = C(j) \langle \alpha' j | \bar{A} | \alpha j \rangle$$

This will be valid also for $\vec{A} = \vec{J}$, with $\alpha' = \alpha$

$\Rightarrow$

$$\langle \alpha j m | J^2 | \alpha j m \rangle = C(j) \langle \alpha j || \vec{J} || \alpha j \rangle$$

but $\langle \alpha j m | J^2 | \alpha j m \rangle = j(j+1)\hbar^2$

$$\Rightarrow \quad C(j) = \frac{j(j+1)\hbar^2}{\langle \alpha j || J || \alpha j \rangle}$$

$$\Rightarrow \quad \langle \alpha' j m | \vec{J}.\vec{A} | \alpha j m \rangle = \frac{j(j+1)\hbar^2 \langle \alpha' j | \vec{A} | \alpha j \rangle}{\langle \alpha j || J | \alpha j \rangle} \qquad (93)$$

using Wigner-Eckart theorem we have

$$\langle \alpha' j m' | A^q | \alpha j m \rangle = \langle j 1 m q | j 1 j m' \rangle \times \langle \alpha' j || \vec{A} | \alpha j \rangle$$

and

$$\langle \alpha j m' | J^q | \alpha j m \rangle = \langle j 1 m q | j 1 j m' \rangle \langle \alpha j || \vec{J} || \alpha j \rangle$$

taking the ratio of these two equations, we have

$$\frac{\langle \alpha' J \| A \| \alpha J \rangle}{\langle \alpha J \| \vec{J} | \alpha J \rangle} = \frac{\langle \alpha' J m' | A^q | \alpha J m \rangle}{\langle \alpha J m' | J^q | \alpha J m \rangle} \quad (94)$$

on substituting (94) in (93) we obtain the desired result

$$\boxed{\langle \alpha' J m' | A^q | \alpha J m \rangle = \frac{\langle \alpha' J m | \vec{J}.\vec{A} | \alpha J m \rangle}{\hbar^2 J(J+1)} \times \langle J m' | J^q | J m \rangle} \quad (95)$$

The importance of projection theorem is that it shows that expectation value $(\alpha = \alpha', m = m')$ of any vector operator is proportional to the expectation value of the angular momentum operator. This implies that any vector associated with a ~~given~~ spherically symmetric quantum mechanical system, will be either parallel or antiparallel to its angular momentum.