EP 222: Classical Mechanics
Tutorial Sheet 1

This tutorial sheet contains problems on the Newton's laws of motion and Lagrangian formalism.

1. Show that for a single particle with a constant mass the equation of motion implies the following differential equation for the kinetic energy:

$$\frac{dT}{dt} = \mathbf{F}\cdot\mathbf{v},$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \mathbf{F}\cdot\mathbf{p}.$$

Soln: (a) We know $T = \frac{1}{2}mv^2 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}$. Thus, if m is constant

$$\frac{dT}{dt} = m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = m\mathbf{a}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v}$$

(b) For a variable mass particle, let us consider

$$\frac{d(mT)}{dt} = \frac{d}{dt}\left(\frac{1}{2}m^2\mathbf{v}\cdot\mathbf{v}\right) = m\frac{dm}{dt}\mathbf{v}\cdot\mathbf{v} + m^2\mathbf{v}\cdot\mathbf{a} = \left(\frac{dm}{dt}\mathbf{v} + m\mathbf{a}\right)\cdot(m\mathbf{v}) = \mathbf{F}\cdot\mathbf{p},$$

because for a variable mass particle $\frac{d\mathbf{p}}{dt} = \frac{dm}{dt}\mathbf{v} + m\mathbf{a}$.

2. Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

Soln: $\mathbf{R}$ is defined as

$$\begin{aligned}
\mathbf{R} &= \frac{\sum_i m_i\mathbf{r}_i}{M} \\
\implies M\mathbf{R} &= \sum_i m_i\mathbf{r}_i \\
\implies M^2R^2 &= (\sum_i m_i\mathbf{r}_i)\cdot(\sum_j m_j\mathbf{r}_j) \\
\implies M^2R^2 &= \sum_{i,j} m_i m_j \mathbf{r}_i\cdot\mathbf{r}_j = \sum_i m_i^2 r_i^2 + \sum_{i\neq j} m_i m_j \mathbf{r}_i\cdot\mathbf{r}_j
\end{aligned}$$

Using

$$r_{ij}^2 = (\mathbf{r}_i - \mathbf{r}_j)^2 = r_i^2 + r_j^2 - 2\mathbf{r}_i \cdot \mathbf{r}_j$$
$$\implies \mathbf{r}_i \cdot \mathbf{r}_j = \frac{1}{2}(r_i^2 + r_j^2 - r_{ij}^2),$$

we obtain

$$\begin{aligned}
M^2R^2 &= \sum_i m_i^2 r_i^2 + \frac{1}{2}\sum_{i\neq j} m_i m_j (r_i^2 + r_j^2 - r_{ij}^2) \\
&= \sum_i m_i^2 r_i^2 + \frac{1}{2}\sum_{i\neq j} m_i m_j r_i^2 + \frac{1}{2}\sum_{i\neq j} m_i m_j r_j^2 + -\frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2 \\
&= \sum_{i\neq j} m_i m_j r_i^2 + \sum_i m_i^2 r_i^2 - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2 \\
&= (\sum_j m_j)(\sum_i m_i r_i^2) - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2 \\
&= M(\sum_i m_i r_i^2) - \frac{1}{2}\sum_{i\neq j} m_i m_j r_{ij}^2 \\
&= M(\sum_i m_i r_i^2) - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2
\end{aligned}$$

In the second sum on the RHS, we can perform unrestricted sum over i and j, because for $i = j$, $r_{ij} = 0$.

3. Show that the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j,$$

can also be written as

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j.$$

These are sometimes called the Nielsen form of Lagrange equations.

Soln: Assuming that

$$T = T(q_i, \dot{q}_i, t),$$

we have

$$\dot{T} = \frac{dT}{dt} = \sum_i \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i + \sum_i \frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial t}$$

so that

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} = \sum_i \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_i}\ddot{q}_i + \sum_i \frac{\partial^2 T}{\partial \dot{q}_j \partial q_i}\dot{q}_i + \frac{\partial T}{\partial q_j} + \frac{\partial^2 T}{\partial \dot{q}_j \partial t}$$

leading to

$$\begin{aligned}
\frac{\partial \dot{T}}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} &= \sum_i \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_i}\ddot{q}_i + \sum_i \frac{\partial^2 T}{\partial \dot{q}_j \partial q_i}\dot{q}_i + \frac{\partial^2 T}{\partial \dot{q}_j \partial t} \\
&= \sum_i \left\{ \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\ddot{q}_i + \frac{\partial}{\partial q_i}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\dot{q}_i \right\} + \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) \\
&= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)
\end{aligned}$$

Substituting this on the LHS of Lagrange equation, we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} &= Q_j \\
\implies \frac{\partial \dot{T}}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - \frac{\partial T}{\partial q_j} &= Q_j \\
\implies \frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} &= Q_j
\end{aligned}$$

4. If L is a Lagrangian for a system of n degrees of freedom satisfying the Lagrange equations, show by direct substitution that

$$L' = L + \frac{dF(q_1, \ldots, q_n, t)}{dt}$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments.

Soln: We have

$$\frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t}.$$

Therefore

$$L' = L + \frac{dF(q_1, \ldots, q_n, t)}{dt} = L + \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t},$$

so that

$$\frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial F}{\partial q_j},$$

leading to

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{d}{dt}\left(\frac{\partial F}{\partial q_j}\right) \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_i \frac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_i + \frac{\partial^2 F}{\partial t \partial q_j} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_i \frac{\partial^2 F}{\partial q_j \partial q_i}\dot{q}_i + \frac{\partial^2 F}{\partial q_j \partial t}
\end{aligned}$$

and

$$\frac{\partial L'}{\partial q_j} = \frac{\partial L}{\partial q_j} + \sum_i \frac{\partial^2 F}{\partial q_j \partial q_i}\dot{q}_i + \frac{\partial^2 F}{\partial q_j \partial t}.$$

Thus

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_i \frac{\partial^2 F}{\partial q_j \partial q_i}\dot{q}_i + \frac{\partial^2 F}{\partial q_j \partial t} - \frac{\partial L}{\partial q_j} - \sum_i \frac{\partial^2 F}{\partial q_j \partial q_i}\dot{q}_i - \frac{\partial^2 F}{\partial q_j \partial t}$$

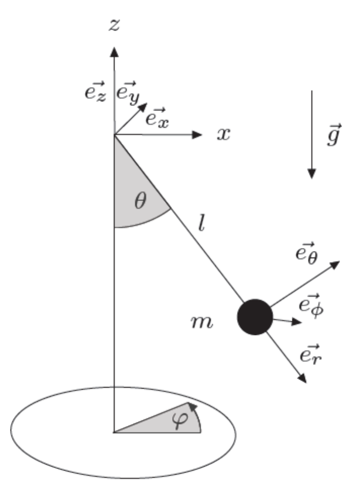
$$\implies \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

QED.

5. Obtain the Lagrange equations of motion for a spherical pendulum, i.e., a point mass suspended by a rigid weightless rod.

Soln: It is best to use spherical polar coordinates here. In the lectures we showed that the kinetic energy of a particle in spherical polar coordinates is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right).$$



If length of the rod is l, then $r = l$, and $\dot{r} = 0$, we are left with two generalized coordinates (θ, ϕ), with

$$T = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2\right),$$

and using the point of suspension as the reference for potential energy, we have

$$V = -mgl\cos\theta.$$

Thus

$$L = T - V = T = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2\right) + mgl\cos\theta.$$

Now the two Lagrange equations are

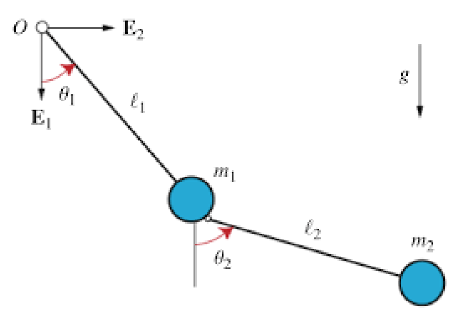
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$
$$\implies ml^2\ddot{\theta} - \frac{1}{2}ml^2\sin 2\theta\dot{\phi}^2 + mgl\sin\theta = 0$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0$$
$$\implies \frac{d(ml^2\sin^2\theta\dot{\phi})}{dt} = 0.$$

6. Obtain the Lagrangian and equations of motion for a double pendulum, where the lengths of the pendula are l_1 and l_2 with corresponding masses m_1 and m_2, confined to move in a plane.

   Soln: As discussed in the lectures, this system has two generalized coordinates θ_1 and θ_2, the angles which the upper and the lower pendula make with respect to the vertical.



   With the motion of the pendula confined in a plane (say, xy plane), then the Cartesian coordinates of the two particles can be written as

$$x_1 = l_1\sin\theta_1$$
$$y_1 = -l_1\cos\theta_1$$

   and

$$x_2 = x_1 + l_2\sin\theta_2 = l_1\sin\theta_1 + l_2\sin\theta_2$$
$$y_2 = y_1 - l_2\cos\theta_2 = -l_1\cos\theta_1 - l_2\cos\theta_2$$

So that

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2).$$

Now

$$\begin{aligned}
\dot{x}_1 &= l_1 \cos\theta_1 \dot{\theta}_1 \\
\dot{y}_1 &= l_1 \sin\theta_1 \dot{\theta}_1 \\
\dot{x}_2 &= l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2 \\
\dot{y}_2 &= l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2
\end{aligned}$$

Easy to verify

$$\begin{aligned}
\dot{x}_1^2 + \dot{y}_1^2 &= l_1^2 \dot{\theta}_1^2, \\
\dot{x}_2^2 + \dot{y}_2^2 &= l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2).
\end{aligned}$$

With this

$$T = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2),$$

and

$$\begin{aligned}
V &= m_1 g y_1 + m_2 g y_2 \\
&= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2,
\end{aligned}$$

so that

$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\
&+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2.
\end{aligned}$$

θ_1 equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0,$$

leads to

$$\frac{d}{dt}\Big((m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\Big) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0.$$

Upon taking the time derivative, we obtain the final form

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 + m_2 l_1 l_2 \sin(\theta_1 - \theta_2) \dot{\theta}_2^2 + (m_1 + m_2) g l_1 \sin\theta_1 \quad = \quad 0$$

For θ_2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0,$$

which yields

$$\frac{d}{dt}\left(m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1-\theta_2)\right) - m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \sin(\theta_1-\theta_2) + m_2 g l_2 \sin\theta_2 = 0.$$

Upon taking the time derivative, we obtain the final form

$$m_2 l_1 l_2 \cos(\theta_1-\theta_2)\ddot{\theta}_1 + m_2 l_2^2\ddot{\theta}_2 - m_2 l_1 l_2 \sin(\theta_1-\theta_2)\dot{\theta}_1^2 + m_2 g l_2 \sin\theta_2 \quad = \quad 0$$

7. If we want to obtain the equations of motion for a charged particle of mass m, moving in an electromagnetic field ($\mathbf{E}$, $\mathbf{B}$), the potential in the Lagrangian has to be velocity dependent $U = q\phi - q\mathbf{A}\cdot\mathbf{v}$, where q is the charge of the particle, and ϕ, and $\mathbf{A}$, respectively, are the scalar and vector potentials of the electromagnetic field so that

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla\times\mathbf{A}.$$

Show that using this Lagrangian, we obtain the correct equations of motion for the particle.

Soln: Using Cartesian coordinates and the fact that $\mathbf{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$, and $\mathbf{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$, we have

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - q\phi + q(A_x\dot{x} + A_y\dot{y} + A_z\dot{z}).$$

Lagrange equation for x component

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$
$$\implies \frac{d}{dt}(m\dot{x} + qA_x) + q\frac{\partial\phi}{\partial x} - q\dot{x}\frac{\partial A_x}{\partial x} - q\dot{y}\frac{\partial A_y}{\partial x} - q\dot{z}\frac{\partial A_z}{\partial x} = 0$$
$$\implies m\ddot{x} = -q\frac{dA_x}{dt} - q\frac{\partial\phi}{\partial x} + q\dot{x}\frac{\partial A_x}{\partial x} + q\dot{y}\frac{\partial A_y}{\partial x} + q\dot{z}\frac{\partial A_z}{\partial x}$$

But

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} + \frac{\partial A_x}{\partial t},$$

so that

$$m\ddot{x} = q\left(-\frac{\partial\phi}{\partial x} - \frac{\partial A_x}{\partial t}\right) - q\frac{\partial A_x}{\partial x}\dot{x} - q\frac{\partial A_x}{\partial y}\dot{y} - q\frac{\partial A_x}{\partial z}\dot{z} + q\dot{x}\frac{\partial A_x}{\partial x} + q\dot{y}\frac{\partial A_y}{\partial x} + q\dot{z}\frac{\partial A_z}{\partial x}$$
$$= q\left(-\frac{\partial\phi}{\partial x} - \frac{\partial A_x}{\partial t}\right) + q\dot{y}\frac{\partial A_y}{\partial x} + q\dot{z}\frac{\partial A_z}{\partial x} - q\frac{\partial A_x}{\partial y}\dot{y} - q\frac{\partial A_x}{\partial z}\dot{z}$$

Using the fact that

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla\times\mathbf{A},$$

we obtain above

$$m\ddot{x} = qE_x + q(\mathbf{v} \times \mathbf{B})_x,$$

which is the x component of the Lorentz force equation. Using the same procedure for y and z components, we obtain

$$m\ddot{\mathbf{r}} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B}).$$

8. The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A} + \nabla\psi(\mathbf{r}, t), \\ \phi &\rightarrow \phi - \frac{\partial \psi}{\partial t}, \end{aligned}$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a moving particle in the electromagnetic field? Is the equation of motion affected?

Soln: On performing the gauge transformations, we have

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A}' = \mathbf{A} + \nabla\psi(\mathbf{r}, t), \\ \phi &\rightarrow \phi' = \phi - \frac{\partial \psi}{\partial t}, \\ L &\rightarrow L' = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\phi' + q\mathbf{A}' \cdot \mathbf{v} = L - q\mathbf{v} \cdot \nabla\psi - q\frac{\partial \psi}{\partial t} \end{aligned}$$

Or

$$\begin{aligned} L' &= L - q\left(\frac{\partial \psi}{\partial x}\dot{x} + \frac{\partial \psi}{\partial y}\dot{y} + \frac{\partial \psi}{\partial z}\dot{z} + \frac{\partial \psi}{\partial t}\right) \\ &= L - q\frac{d\psi}{dt} \end{aligned}$$

Because L' differs from L by the total time derivative of a differentiable function $\psi = \psi(\mathbf{r}, t)$, hence, from the result of Prob 4, it will lead to the same Lagrange equations as L.