EP 222: Classical Mechanics
Tutorial Sheet 2: Solution

This tutorial sheet contains problems related to the calculus of variations, and Hamilton's principle.

1. Show that the geodesics of a spherical surface are great circles, i.e., circles whose centers lie at the center of the sphere.

Soln: Length element on the surface of a sphere of radius a is

$$ds = a\sqrt{d\theta^2 + \sin^2\theta d\phi^2},$$

So the distance between two points 1 and 2

$$S = \int_1^2 ds = a\int_{\theta_1}^{\theta_2} \sqrt{1+\sin^2\theta\phi'^2}d\theta \tag{1}$$

where $\phi' = \frac{d\phi}{d\theta}$. The fact that the integral of equation (1) is a minimum, implies the Euler-Lagrange equations :

$$\frac{d}{d\theta}\left(\frac{\partial I}{\partial\dot{\phi}}\right) - \frac{\partial I}{\partial\phi} = 0,$$

$$\frac{d}{d\theta}\left\{\frac{\dot{\phi}\sin^2\theta}{\sqrt{1+\sin^2\theta\phi'^2}}\right\} = 0,$$

$$\left\{\frac{\dot{\phi}\sin^2\theta}{\sqrt{1+\sin^2\theta\phi'^2}}\right\} = b = \text{constant} \ ,$$

$$\dot{\phi}^2\sin^4\theta = b^2 + b^2\sin^2\theta\phi'^2,$$

$$\frac{d\phi}{d\theta} = \frac{b}{\sin\theta\sqrt{\sin^2\theta - b^2}},$$

$$\phi = b\int \frac{\csc^2\theta}{\sqrt{1-b^2\csc^2\theta}}d\theta + C,$$

using, $1+\cot^2\theta = \csc^2\theta$, we get,

$$\phi = b\int \frac{\csc^2\theta}{\sqrt{1-b^2\csc^2\theta}}d\theta + C,$$

$$\phi = \int \frac{\csc^2\theta}{\sqrt{\frac{(1-b^2)}{b^2} - \cot^2\theta}}d\theta + C,$$

Let, $d^2 = \frac{(1-b^2)}{b^2}$, and $t = \cot\theta$, $dt = -\csc^2\theta d\theta$,

$$\phi = -\int \frac{dt}{\sqrt{d^2-t^2}} + C,$$

$$\phi = -\sin^{-1}\left(\frac{t}{d}\right) + C,$$

$$-t = d\sin(\phi - C),$$
$$-\frac{\cos\theta}{\sin\theta} = d(\sin\phi\cos C - \cos\phi\sin C),$$
$$-\cos\theta = A\sin\theta\sin\phi - B\sin\theta\cos\phi, \tag{2}$$

above, $A = d\cos C$,$B = d\sin C$. Multiply both sides of equation (2) by radius a, and recognize,

$$a\sin\theta\sin\phi = y,$$

$$a\sin\theta\cos\phi = x,$$

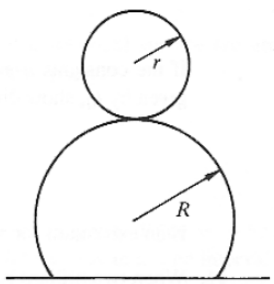
$$a\cos\theta = z,$$

we get above,

$$Bx - Ay - z = 0 \tag{3}$$

So, the shortest length curve on the surface of a sphere lies on its intersection with a plane which passes through the origin of the sphere (Eq. (3)). This is the definition of a great circle.

2. A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R. The only external force is that of gravity. If the smaller cylinder starts rolling from rest on top of the bigger cylinder, use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder.



Soln: For the most general motion, we need three generalized coordinate,

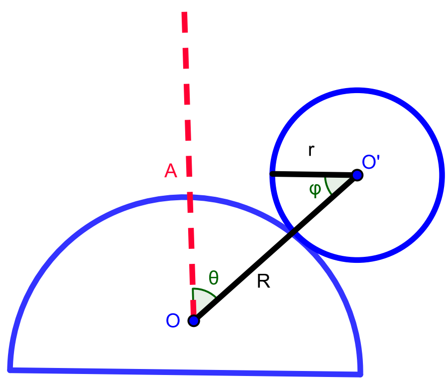


Figure 1:

$a =$ Distance between the center of the hoop and the center of the cylinder$=$ OO'. $\theta =$ Angle made by OO' from OA direction. This describes the motion of the c.m. of the hoop along the surface of the cylinder. Angle ϕ, as shown above, defines the rotation of the hoop about its center. Clearly,

$$T = \frac{1}{2}m\dot{a}^2 + \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}I_{hoop}\dot{\phi}^2,$$

Since, $I_{hoop} = mr^2$,

$$T = \frac{1}{2}m\dot{a}^2 + \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\phi}^2,$$

and, $V = mga\cos\theta$, so,

$$L = \frac{1}{2}m\dot{a}^2 + \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - mga\cos\theta$$

However, the motion of the hoop has constraints: (1) It moves along the surface of cylinder,

$$a = r + R,$$

$$da = 0,$$

(2) Hoop rolls without slipping,

$$ad\theta = rd\phi,$$

$$(r + R)\,d\theta - rd\phi = 0,$$

We will have to introduce two Lagrange multipliers λ_1 (for a) and λ_2 (for θ and ϕ).

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right) - \frac{\partial L}{\partial a} = \lambda_1,$$

$$m\ddot{a} + mg\cos\theta - ma\dot{\theta}^2 = \lambda_1, \tag{4}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = (r + R)\,\lambda_2,$$

$$\frac{d}{dt}\left(ma^2\dot{\theta}\right) - mga\sin\theta = (r + R)\,\lambda_2, \tag{5}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = -r\lambda_2,$$

$$mr^2\ddot{\phi} = -r\lambda_2, \tag{6}$$

If in (4), (5) and (6) we substitute the constraint equations,

$$a = r + R,$$

$$\dot{a} = \ddot{a} = 0,$$

$$\ddot{\phi} = \left(\frac{r + R}{r}\right)\ddot{\theta},$$

we get,

$$\lambda_1 = mg\cos\theta - m\,(r + R)\,\theta^2,$$

$$\left(m\left(r+R\right)^2\ddot{\theta}\right)-mg\left(r+R\right)\sin\theta=\left(r+R\right)\lambda_2,$$

$$mr\left(r+R\right)\ddot{\theta}=-r\lambda_2,$$

The point where hoops leaves the cylinder, reaction force λ_1 should vanish

$$mg\cos\theta-m\left(r+R\right)\dot{\theta}^2=0. \tag{7}$$

However, $\dot{\theta}$ at that point can be found by conservation of energy (gain in K.E.= loss in P.E.),

$$\frac{1}{2}m\left(r+R\right)^2\dot{\theta}^2+\frac{1}{2}mr^2\dot{\phi}^2=mg\left(r+R\right)\left(1-\cos\theta\right),$$

with no slipping constraint,

$$\frac{1}{2}m\left(r+R\right)^2\dot{\theta}^2+\frac{1}{2}mr^2\left(\frac{r+R}{r}\right)^2\dot{\theta}^2=mg\left(r+R\right)\left(1-\cos\theta\right),$$

$$m\left(r+R\right)^2\dot{\theta}^2=mg\left(r+R\right)\left(1-\cos\theta\right),$$

$$m\left(r+R\right)\dot{\theta}^2=mg\left(1-\cos\theta\right), \tag{8}$$

putting equation (8) in (7) to get,

$$mg\cos\theta-mg\left(1-\cos\theta\right)=0,$$

$$\cos\theta=\frac{1}{2},$$

$$\theta=60^\circ.$$

So, hoop will fall out at $\theta=60^\circ$.

3. A point mass is constrained to move on a massless hoop of radius a fixed in a vertical plane that rotates about its vertical symmetry axis with constant angular speed ω. Obtain the Lagrange equations of motion assuming the only external forces arise from gravity. What are the constants of motion? Show that if ω is greater than a critical value ω_0, there can be a solution in which the particle remains stationary on the hoop at a point other than at the bottom, but that if $\omega<\omega_0$, the only stationary point for the particle is at the bottom of the hoop. What is the value of ω_0?

Soln: Here, we have two generalized coordinates θ and ϕ. Thus the Lagrangian is,

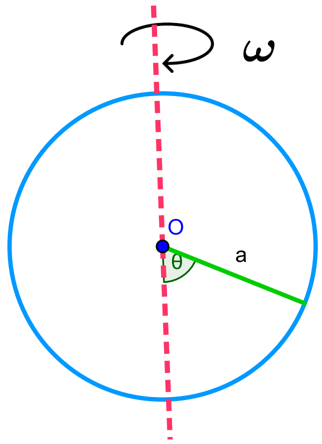


Figure 2:

$$L = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mga\cos\theta,$$

But the motion in ϕ is trivial i.e.

$$\dot{\phi} = \omega = \text{constant}.$$

So, the effective Lagrangian has only one generalized coordinate θ

$$L = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\omega^2\right) + mga\cos\theta,$$

and the equation of motion is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0,$$

$$ma^2\ddot{\theta}^2 - ma^2\sin\theta\cos\theta\omega^2 + mga\sin\theta = 0, \tag{9}$$

Since, the Lagrangian is time independent energy function will be a constant, which can be obtained by multiplying equation (9) by $\dot{\theta}$ in both sides,

$$ma^2\dot{\theta}\ddot{\theta}^2 - ma^2\sin\theta\cos\theta\omega^2\dot{\theta} + mga\sin\theta\dot{\theta} = 0,$$

$$\frac{d}{dt}\left(\frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{4}ma^2\cos 2\theta\omega^2 - mga\cos\theta\right) = 0,$$

$$\frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{4}ma^2\cos 2\theta\omega^2 - mga\cos\theta = c,$$

at $t = 0$, $\theta = \dot{\theta} = 0$ (assume), we obtain

$$c = \frac{1}{4}ma^2\omega^2 - mga.$$

With this,

$$\frac{1}{2}ma^2\dot{\theta}^2 = \frac{1}{4}ma^2\left(1 - \cos 2\theta\right)\omega^2 - mga\left(1 - \cos\theta\right),$$

We are looking for solution $\dot{\theta} = 0$,

$$\frac{1}{4}ma^2 \left(1 - \cos 2\theta\right) \omega^2 - mga \left(1 - \cos\theta\right) = 0,$$

$$\frac{1}{4}ma^2\omega^2 2\sin^2\theta - 2mga\sin^2\frac{\theta}{2} = 0,$$

$$2ma^2\omega^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} - 2mga\sin^2\frac{\theta}{2} = 0,$$

$$\sin^2\frac{\theta}{2}\left(\cos^2\frac{\theta}{2} - \frac{g}{a\omega^2}\right) = 0,$$

$$\implies \sin^2\frac{\theta}{2} = 0,$$

$$\implies \theta = 0,$$

or,

$$\cos^2\frac{\theta}{2} = \frac{g}{a\omega^2}.$$

we know that

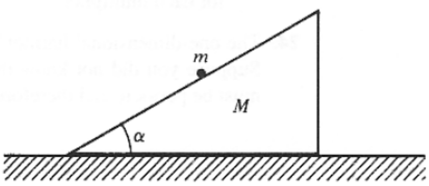
$$\cos^2\frac{\theta}{2} \leq 1,$$

$$\frac{g}{a\omega^2} \leq 1,$$

$$\omega^2 \geq \frac{g}{a} = \omega_0^2.$$

So, if $\omega < \omega_0$, the only stationary point is at the bottom. But for $\omega \geq \omega_0$, we will have stationary point for $\theta > 0$.

4. A particle of mass m slides without friction on a wedge of angle α and mass M that can move without friction on a smooth horizontal surface, as shown in the figure. Treating the constraint of the particle on the wedge by the method of Lagrange multipliers, find the equation of motion for particle and wedge. Also obtain an expression for the forces of constraint. Calculate the work done in time t by the forces of constraint acting on the particle and on the wedge. What are the constants of motion for the system?



Soln: We can describe the rotation of the wedge w.r.t a coordinate system fixed in the ground $(X - Y)$ and that of the particle w.r.t a coordinate system fixed on the wedge $(x - y)$. The kinetic energy of the wedge can be written in terms of the $(X - Y)$ coordinated of an point, say O', on the wedge.

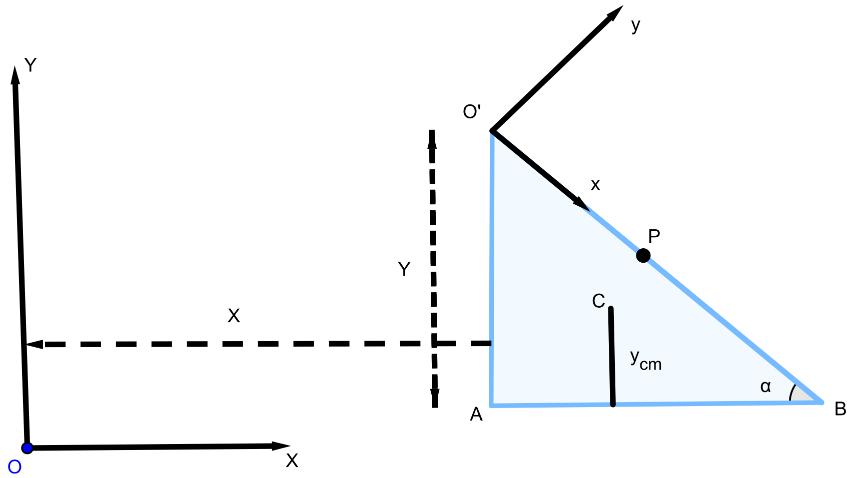


Figure 3:

$$T_{wedge} = \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2\right),$$

where (X, Y) are the coordinate of O' w.r.t the fixed frame. If X_p and Y_p are the coordinates of m w.r.t the fixed frame we get the kinetic energy of the mass m,

$$T_m = \frac{1}{2}m\left(\dot{X}_p^2 + \dot{Y}_p^2\right),$$

so, the Lagrangian is,

$$L = \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{1}{2}m\left(\dot{X}_p^2 + \dot{Y}_p^2\right) - mgY_p - Mg\left(Y - h\right), \tag{10}$$

here, $(Y - h)$ is the Y i c.m coordinate of wedge. But from the geometry it is clear,

$$X_p = X + x\cos\alpha,$$

$$Y_p = Y - x\sin\alpha,$$

Substituting X_p in equation (10)

$$L = \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{1}{2}m\left(\left(\dot{X} + \dot{x}\cos\alpha\right)^2 + \dot{Y}_p^2\right) - mgY_p - Mg\left(Y - h\right),$$

$$L = \frac{1}{2}M\left(\dot{X}^2 + \dot{x}^2\right) + \frac{1}{2}m\dot{x}^2\cos\alpha + m\dot{X}\dot{x}\cos\alpha + \frac{1}{2}m\dot{Y}_p^2 + \frac{1}{2}M\dot{Y}^2 - mgY_p - Mg\left(Y - h\right),$$

This Lagrangian is expressed in four generalized coordinates X, x, Y and Y_p. But, Here we have two constraints: (1) wedge is moving only along the $X-$direction,

$$dY = 0, \tag{11}$$

(2) particle is moving along the wedge in the $x-$direction. So the constraint equation is,

$$Y_p = Y - x\sin\alpha,$$

$$dY_p = dY - x\sin\alpha dx,$$

$$dY_p - dY + x \sin\alpha dx = 0. \tag{12}$$

So, we introduce Lagrange multipliers corresponding to these constraints. Let λ_1 and λ_2 corresponds to equation (11) and (12), respectively and we have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) - \frac{\partial L}{\partial X} = 0,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}}\right) - \frac{\partial L}{\partial Y} = \lambda_1 - \lambda_2,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \sin\alpha\lambda_2,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}_p}\right) - \frac{\partial L}{\partial Y_p} = \lambda_2,$$

so, we get,

$$(m+M)\ddot{X} + m\ddot{x}\cos\alpha = 0,$$

$$M\ddot{Y} + Mg = \lambda_1 - \lambda_2, \tag{13}$$

$$m\ddot{x}\cos^2\alpha + m\ddot{X}\cos\alpha = \sin\alpha\lambda_2,$$

$$m\ddot{Y}_p + mg = \lambda_2, \tag{14}$$

the constraints of motion imply,

$$\ddot{Y} = 0,$$

and

$$\ddot{Y}_p = -\ddot{x}\sin\alpha,$$

So, we get from (13) and (14),

$$\lambda_1 - \lambda_2 = Mg, \tag{15}$$

$$mg - m\ddot{x}\sin\alpha = \lambda_2, \tag{16}$$

and from (15) and (16),

$$\lambda_1 = \lambda_2 + Mg = (m+M)g - m\ddot{x}\sin\alpha, \tag{17}$$

with these constraints we have only two equations of motion left,

$$(m+M)\ddot{X} + m\ddot{x}\cos\alpha = 0,$$

$$m\ddot{x}\cos^2\alpha + m\ddot{X}\cos\alpha = \sin\alpha\,(mg - m\ddot{x}\sin\alpha),$$

or we get,

$$(m+M)\ddot{X} + m\ddot{x}\cos\alpha = 0, \tag{18}$$

$$m\ddot{X}\cos\alpha + m\ddot{x} = mg\sin\alpha, \tag{19}$$

from equations (18) and (19) we get separate equations of motions for $\ddot{X}$ and $\ddot{x}$,

$$\left(m\sin^2\alpha + M\right)\ddot{X} = -mg\sin\alpha\cos\alpha,$$

and

$$m\ddot{x} = \frac{mg\sin\alpha\,(m+M)}{M + m\sin^2\alpha},$$

leading to,

$$\ddot{X} = \frac{-mg\sin\alpha\cos\alpha}{\left(m\sin^2\alpha + M\right)}, \tag{20}$$

$$\ddot{x} = \frac{g\sin\alpha\,(m+M)}{M + m\sin^2\alpha}. \tag{21}$$

Let us substitute equation (20) and (21) in equation (16) and (17) to get λ_1 and λ_2,

$$\lambda_1 = (m+M)\,g - \frac{mg\sin^2\alpha\,(m+M)}{M + m\sin^2\alpha},$$

or

$$\lambda_1 = \frac{M\,(m+M)\,g}{M + m\sin^2\alpha},$$

and

$$\lambda_2 = \frac{Mmg\cos^2\alpha}{M + m\sin^2\alpha}.$$

Constants of Motion : (1) Since Lagrangian is not an explicit function of time, so the energy function h will be a constant of motion.
(2) From equation (18), we get

$$\frac{d}{dt}\left((m+M)\,\dot{X} + m\dot{x}\cos\alpha = 0\right),$$

$$(m+M)\,\dot{X} + m\dot{x}\cos\alpha = \text{constant},$$

$X-$component of the momentum of the whole system (particle + wedge) is constant. This is obvious because there is no force in the $x-$direction on the system, hence momentum in that direction should be conserved.

5. The one-dimensional harmonic oscillator has the Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$. Suppose you did not know the solution to the motion but realized that the motion must be periodic and therefore could be described by a Fourier series of the form

$$x(t) = \sum_{j=0} a_j\cos j\omega t,$$

(taking $t = 0$ at a turning point) where ω is the unknown angular frequency of the motion. This representation for $x(t)$ defines a many-parameter path for the system point in configuration space. Consider the action integral I for two points t_1 and t_2 separated by the period $T = \frac{2\pi}{\omega}$. Show that with this form for the system path, I is extremum for nonvanishing x only if $a_j = 0$, for all $j \neq 1$, and only if $\omega^2 = k/m$.

Soln: The Lagrangian of one-dimensional harmonic oscillator is,

$$L = \frac{1}{2}m\dot{x^2} - \frac{1}{2}kx^2,$$

and it is given that motion can be describe by Fourier series,

$$x(t) = \sum_{j=0}^{\infty} a_j \cos j\omega t,$$

$$\dot{x}(t) = -\sum_{j=0}^{\infty} j\omega a_j \sin j\omega t,$$

and

$$(\dot{x}(t))^2 = \sum_{i,j=0}^{\infty} ij\omega^2 a_i a_j \sin i\omega t \sin j\omega t,$$

$$(x(t))^2 = \sum_{i,j=0}^{\infty} a_i a_j \cos i\omega t \cos j\omega t.$$

So, the action integral between t_1 and $\left(t_1 + \frac{2\pi}{\omega}\right)$, will be,

$$I = \int_{t_1}^{\left(t_1+\frac{2\pi}{\omega}\right)} L dt,$$

$$= \int_{t_1}^{\left(t_1+\frac{2\pi}{\omega}\right)} \left\{ \frac{1}{2}m \sum_{i,j=0}^{\infty} ij\omega^2 a_i a_j \sin i\omega t \sin j\omega t - \frac{1}{2}k \sum_{i,j=0}^{\infty} a_i a_j \cos i\omega t \cos j\omega t \right\} dt,$$

$$= \frac{1}{2}m \sum_{i,j=0}^{\infty} ij\omega^2 a_i a_j \int_{t_1}^{\left(t_1+\frac{2\pi}{\omega}\right)} \sin i\omega t \sin j\omega t\, dt - \frac{1}{2}k \sum_{i,j=0}^{\infty} a_i a_j \int_{t_1}^{\left(t_1+\frac{2\pi}{\omega}\right)} \cos i\omega t \cos j\omega t\, dt, \quad (22)$$

But we can show,

$$\int_{t_1}^{\left(t_1+\frac{2\pi}{\omega}\right)} \sin i\omega t \sin j\omega t\, dt = \int_{t_1}^{\left(t_1+\frac{2\pi}{\omega}\right)} \cos i\omega t \cos j\omega t\, dt = \frac{\pi}{\omega}\delta_{i,j}. \quad (23)$$

So we get after substituting equation (23) in (22),

$$I = \frac{1}{2}m \sum_{i,j=0}^{\infty} ij\omega^2 a_i a_j \frac{\pi}{\omega}\delta_{ij} - \frac{1}{2}k \sum_{i,j=0}^{\infty} a_i a_j \frac{\pi}{\omega}\delta_{ij},$$

$$I = \frac{\pi}{2\omega} \sum_{j=0}^{\infty} a_j^2 \left(j^2 m\omega^2 - k\right). \quad (24)$$

If $x(t)$ is the correct solution, variation of (24) with respect to a_j's should be satisfactory,

$$\delta I = \frac{\pi}{2\omega} \sum_{j=0}^{\infty} 2a_j \left(j^2 m\omega^2 - k\right) \delta a_j = 0,$$

or

$$a_0 k \delta a_0 + \sum_{j=1}^{\infty} 2a_j \left(j^2 m\omega^2 - k\right) \delta a_j = 0.$$

Since all the variation of $\delta a_j$are independent of each other, their coefficients should vanish,

$$a_0 k = 0, \tag{25}$$

$$a_j \left(j^2 m\omega^2 - k\right) = 0, \text{for } j = 1, 2, 3, ... \tag{26}$$

Clearly from equation (25), we conclude that

$$a_0 = 0,$$

because

$$k \neq 0.$$

Clearly, all but one a_j (for $j > 0$) above will vanish because, otherwise $(j^2 m\omega^2 - k) = 0$ for non-vanishing a_j's leading to more than one value of frequency. If we choose $a_j = 0$, for $j = 1$, we get from equation (26)

$$m\omega^2 - k = 0,$$

$$\omega = \sqrt{\frac{k}{m}}.$$

Actually the book leads to a false impression. You can have any one $a_j \neq 0$, and the final solution will be the same as above, as shown in the tutorial. Try it !!!!!