EP 222: Classical Mechanics
Tutorial Sheet 5

This tutorial sheet contains problems related to angular momentum, inertia tensor, and rigid body motion.

1. Three equal point masses m are located at $(a, 0, 0)$, $(0, a, 2a)$, and $(0, 2a, a)$. Find the principal moments of inertia about the origin and a set of principal axes.

2. Obtain the inertia tensor of a system, consisting of four identical particles of mass m each, arranged on the vertices of a square of sides of length $2a$, with the coordinates of the four particles given by $(\pm a, \pm a, 0)$.

3. A rigid body consists of three point masses of 2 kg, 1 kg, and 4 kg, connected by massless rods. These masses are located at coordinates (1, -1,1), (2,0,2), and (-1,1,0) in meters, respectively. Compute the inertia tensor of this system. What is the angular momentum vector of this body, if it is rotating with an angular velociy $\boldsymbol{\omega} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$?

4. Obtain the moment of inertia tensor of a thin uniform rod of length l, and mass M, assuming that the origin of the coordinate system is at the center of mass of the rod.

5. Obtain the moment of inertia tensor of a thin uniform ring of radius R, and mass M, with the origin of the coordinate system placed at the center of the ring, and the ring lying in the xy plane.

6. Obtain the moment of inertia tensor of a thin uniform disk of radius R, and mass M, with the origin of the coordinate system placed at the center of the disk, and the disk lying in the xy plane.

7. Obtain the moment of inertia tensor of a uniform solid sphere of radius R, and mass M, with the origin of the coordinate system placed at the center of the sphere. Note that this problem can be done by dividing the sphere into a large number of infinitesimally thin disks.

8. Obtain the moment of inertia tensor of a uniform hollow sphere of radius R, and mass M, with the origin of the coordinate system placed at the center of the sphere. Note that this problem can be done by dividing the sphere into a large number of infinitesimally thin rings.

9. Consider an asymmetric rigid body with principal moments of inertia $I_1 \neq I_2 \neq I_3$. Assuming that it is initially rotating about the principle axis $\hat{\mathbf{e}}_1$, with angular velocity ω_1, without any external torque. Suddenly, an external torque is applied to it for a brief time. Using the Euler equations show that

   (a) If I_1 is the smallest or the largest of I_1, I_2, I_3, the rotation of the rigid body will continue about the $\hat{\mathbf{e}}_1$ axis, in a stable manner

   (b) Otherwise, if the value of I_1 is intermediate compared to I_2 and I_3, then after the application of the external torque, the body will spin out of control.

10. Consider a rigid body with cylindrical symmetry so that its moments of inertia with respect to the principle axes are $I_1 = I$, $I_2 = I_3 = I_\perp$. If this body is rotating about a general axis without any external torque, write down its Euler's equations, and solve them.

11. In the lectures we proved the following result about the angular momentum of a rigid body of mass M

$$\mathbf{L} = \mathbf{R} \times (M\mathbf{V}_{cm}) + \sum_i m_i \mathbf{r}'_i \times \dot{\mathbf{r}}'_i,$$

above $\mathbf{V}_{cm}$ is the velocity of the center of mass, $\mathbf{R}$ its location, while $\mathbf{r}'_i$ and $\dot{\mathbf{r}}'_i$ are positions and velocities, respectively of the i-th particle of the rigid body w.r.t. to its center of mass. Using this equation show that

(a)

$$L_z = I_0\omega + M(\mathbf{R} \times \mathbf{V}_{cm})_z,$$

where I_0 is the moment of inertia of the body about z-axis passing through the center of mass of the body.

(b) kinetic energy of the rigid body also splits in similar two terms, and can be written as

$$K = \frac{1}{2}I_0\omega^2 + \frac{1}{2}MV_{cm}^2$$

(c) work-energy theorem holds for the rotational motion

$$\int_{\theta_a}^{\theta_b} \tau_0 d\theta = \frac{1}{2}I_0\omega_b^2 - \frac{1}{2}I_0\omega_b^2,$$

where τ_0 is the torque acting on the rigid body, while subscript a and b denote initial and final quantities, respectively.

12. Prove the following results about the rotational kinetic energy $K_{rot} = \frac{1}{2}\sum_i m_i \dot{\mathbf{r}}'^2_i$ of a general rigid body

(a)

$$K_{rot} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}$$

(b)

$$K_{rot} = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3},$$

where L_i and I_i are the angular momentum component, and moment of inertia, respectively, with respect to the i-th principal axis.