EP 222: Classical Mechanics
Tutorial Sheet 8: Solution

This tutorial sheet contains problems related to canonical transformations, Poisson brackets etc.

1. One of the attempts at combining two sets of Hamilton's equations into one tries to take q and p as forming a complex quantity. Show directly from Hamilton's equations of motion that for a system of one degree of freedom the transformation

$$Q = q + ip, \qquad P = Q^*$$

is not canonical if the Hamiltonian is left unaltered. Can you find another set of coordinates Q' and P' that are related to Q, P by a change of scale only, and that are canonical?

Soln: A given transformation is canonical if the Hamilton's equations are satisfied in the transformed coordinate system. Therefore, let us evaluate $\frac{\partial H}{\partial Q}$ and $\frac{\partial H}{\partial P}$

$$\frac{\partial H}{\partial Q} = \frac{\partial H}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial Q}$$
$$\frac{\partial H}{\partial P} = \frac{\partial H}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial P}$$

Using the fact that canonical variables (q, p) satisfy Hamilton's equations, we obtain

$$\frac{\partial H}{\partial Q} = -\dot{p}\frac{\partial q}{\partial Q} + \dot{q}\frac{\partial p}{\partial Q}$$
$$\frac{\partial H}{\partial P} = -\dot{p}\frac{\partial q}{\partial P} + \dot{q}\frac{\partial p}{\partial P}$$

Given the fact that

$$q = \frac{1}{2}(P + Q)$$
$$p = \frac{i}{2}(P - Q),$$

we have

$$\frac{\partial q}{\partial Q} = \frac{\partial q}{\partial P} = \frac{1}{2}$$
$$\frac{\partial p}{\partial Q} = -\frac{\partial p}{\partial P} = -\frac{i}{2}$$

Substituting these above, we obtain

$$\frac{\partial H}{\partial Q} = -\frac{1}{2}\dot{p} - \frac{i}{2}\dot{q} = -\frac{i}{2}(\dot{q} - i\dot{p}) = -\frac{i}{2}\dot{P}$$
$$\frac{\partial H}{\partial P} = -\frac{1}{2}\dot{p} + \frac{i}{2}\dot{q} = \frac{i}{2}(\dot{q} + i\dot{p}) = \frac{i}{2}\dot{Q}$$

Thus, Hamiltonian H expressed in terms of Q and P does not satisfy the Hamilton's equations, making the transformation non-canonical. Let us scale these variables to define $Q' = \lambda Q$, and $P' = \mu P$, so that

$$\frac{\partial H}{\partial Q'} = \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial Q'} = -\frac{i\dot{P}}{2\lambda} = -\frac{i}{2\lambda\mu}\dot{P}'$$
$$\frac{\partial H}{\partial P'} = \frac{\partial H}{\partial P}\frac{\partial P}{\partial P'} = \frac{i\dot{Q}}{2\mu} = \frac{i}{2\lambda\mu}\dot{Q}'.$$

If we choose λ and μ such that $\lambda\mu = \frac{i}{2}$, the Hamilton's equations will be satisfied in variables Q' and P' , and the transformation will become canonical. One choice which will achieve that is

$$\lambda = \mu = \frac{i^{1/2}}{\sqrt{2}} = \frac{(1+i)}{2}$$

2. Show that the transformation for a system of one degree of freedom,

$$Q = q\cos\alpha - p\sin\alpha$$
$$P = q\sin\alpha + p\cos\alpha,$$

satisfies the symplectic condition for any value of the parameter α. Find a generating function for the transformation. What is the physical significance of the transformation for $\alpha = 0$? For $\alpha = \pi/2$? Does your generating function work for both the cases?
Soln: We will check the symplectic conditions using the order of variables

$$\eta = \begin{pmatrix} q \\ p \end{pmatrix}$$
$$\zeta = \begin{pmatrix} Q \\ P \end{pmatrix},$$

with this

$$M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$$
$$= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Now we check the two symplectic conditions

$$M^TJM = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$
$$= \begin{pmatrix} \sin\alpha\cos\alpha - \sin\alpha\cos\alpha & \sin^2\alpha + \cos^2\alpha \\ -\sin^2\alpha - \cos^2\alpha & \sin\alpha\cos\alpha - \sin\alpha\cos\alpha \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$$

Thus, symplectic condition 1 is satisfied. Similarly, it is easy to verify that the second symplectic condition $MJM^T = J$ is also satisfied for all values of α, making the transformation canonical. Let us try to find a generating function of the first type, i.e., $F_1(q, Q)$ for the transformation. The governing equations for F_1 are

$$p = \frac{\partial F_1}{\partial q}$$
$$P = -\frac{\partial F_1}{\partial Q}$$

Using the transformation equations, we can express both p and P in terms of q and Q, as follows

$$p = q \cot\alpha - Q \csc\alpha$$
$$P = q \sin\alpha + p \cos\alpha = q \sin\alpha + (q \cot\alpha - Q \csc\alpha)\cos\alpha$$
$$\implies P = q\left(\frac{\cos^2\alpha}{\sin\alpha} + \sin\alpha\right) - Q \cot\alpha = q \csc\alpha - Q \cot\alpha.$$

Now we integrate the generating equations

$$\frac{\partial F_1}{\partial q} = p = q \cot\alpha - Q \csc\alpha$$
$$\implies F_1 = \frac{q^2}{2} \cot\alpha - Qq \csc\alpha + f(Q).$$

Using this in the second generating equation for F_1, $\frac{\partial F_1}{\partial Q} = -P$, we obtain

$$-q \csc\alpha + \frac{df}{dQ} = -q \csc\alpha + Q \cot\alpha$$
$$\implies \frac{df}{dQ} = Q \cot\alpha$$
$$\implies f(Q) = \frac{Q^2}{2} \cot\alpha,$$

leading to the final expression for generating function

$$F_1(q, Q) = \frac{1}{2}\left(q^2 + Q^2\right) \cot\alpha - Qq \csc\alpha.$$

Let us consider $\alpha = 0$, which is nothing but the identity transformation, and our F_1 is indeterminate for that case. This is understandable because we know that this transformation is generated by $F_2 = qP$. We would have got the correct limiting behavior for this case if we had instead used F_2 generating function. For $\alpha = \pi/2$, we have the interchange transformation, and our generating function becomes $F_1 = -qQ$, which is the correct result.

3. Show directly that the transformation

$$Q = \log\left(\frac{1}{q} \sin p\right), \qquad P = q \cot p$$

is canonical.
Soln: We need to just check one of the symplectic conditions, with

$$\begin{aligned} M &= \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\csc^2 p \end{pmatrix}. \end{aligned}$$

Now we check the symplectic condition

$$\begin{aligned} M^T JM &= \begin{pmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\csc^2 p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\csc^2 p \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\csc^2 p \end{pmatrix} \begin{pmatrix} \cot p & -q\csc^2 p \\ \frac{1}{q} & -\cot p \end{pmatrix} \\ &= \begin{pmatrix} \frac{\cot p}{q} - \frac{\cot p}{q} & \csc^2 p - \cot^2 p \\ -(\csc^2 p - \cot^2 p) & -q\csc^2 p\cot p + q\csc^2 p\cot p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J \end{aligned}$$

Because the symplectic condition is satisfied, the transformation is canonical.

4. Show directly that for a system of one degree of freedom the transformation

$$Q = \arctan\frac{\alpha q}{p}, \qquad P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

is canonical, where α is an arbitrary constant of suitable dimensions.
Soln: We will just check one of the symplectic conditions, with

$$\begin{aligned} M &= \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\alpha p}{p^2+\alpha^2 q^2} & -\frac{\alpha q}{p^2+\alpha^2 q^2} \\ \alpha q & \frac{p}{\alpha} \end{pmatrix}. \end{aligned}$$

Let us check the symplectic condition

$$\begin{aligned} M^T JM &= \begin{pmatrix} \frac{\alpha p}{p^2+\alpha^2 q^2} & \alpha q \\ -\frac{\alpha q}{p^2+\alpha^2 q^2} & \frac{p}{\alpha} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\alpha p}{p^2+\alpha^2 q^2} & -\frac{\alpha q}{p^2+\alpha^2 q^2} \\ \alpha q & \frac{p}{\alpha} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\alpha p}{p^2+\alpha^2 q^2} & \alpha q \\ -\frac{\alpha q}{p^2+\alpha^2 q^2} & \frac{p}{\alpha} \end{pmatrix} \begin{pmatrix} \alpha q & p/\alpha \\ -\frac{\alpha p}{p^2+\alpha^2 q^2} & \frac{\alpha q}{p^2+\alpha^2 q^2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\alpha^2 pq - \alpha^2 pq}{p^2+\alpha^2 q^2} & \frac{p^2+\alpha^2 q^2}{p^2+\alpha^2 q^2} \\ -\frac{p^2+\alpha^2 q^2}{p^2+\alpha^2 q^2} & \frac{pq-pq}{p^2+\alpha^2 q^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J. \end{aligned}$$

Thus the transformation is canonical.

5. The transformation between two sets of coordinates are

$$\begin{aligned} Q &= \log(1 + q^{1/2}\cos p), \\ P &= 2(1 + q^{1/2}\cos p)q^{1/2}\sin p. \end{aligned}$$

(a) Show directly from these transformation equations that Q, P are canonical variables if q and p are.

Soln: We will just check one of the symplectic conditions, with

$$M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\cos p}{2q^{1/2}(1+q^{1/2}\cos p)} & -\frac{q^{1/2}\sin p}{(1+q^{1/2}\cos p)} \\ \frac{(1+2q^{1/2}\cos p)\sin p}{q^{1/2}} & 2q^{1/2}(\cos p + q^{1/2}\cos 2p) \end{pmatrix},$$

so that

$$M^T J M = \begin{pmatrix} \frac{\cos p}{2q^{1/2}(1+q^{1/2}\cos p)} & \frac{(1+2q^{1/2}\cos p)\sin p}{q^{1/2}} \\ -\frac{q^{1/2}\sin p}{(1+q^{1/2}\cos p)} & 2q^{1/2}(\cos p + q^{1/2}\cos 2p) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{\cos p}{2q^{1/2}(1+q^{1/2}\cos p)} & -\frac{q^{1/2}\sin p}{(1+q^{1/2}\cos p)} \\ \frac{(1+2q^{1/2}\cos p)\sin p}{q^{1/2}} & 2q^{1/2}(\cos p + q^{1/2}\cos 2p) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\cos p}{2q^{1/2}(1+q^{1/2}\cos p)} & \frac{(1+2q^{1/2}\cos p)\sin p}{q^{1/2}} \\ -\frac{q^{1/2}\sin p}{(1+q^{1/2}\cos p)} & 2q^{1/2}(\cos p + q^{1/2}\cos 2p) \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{(1+2q^{1/2}\cos p)\sin p}{q^{1/2}} & 2q^{1/2}(\cos p + q^{1/2}\cos 2p) \\ -\frac{\cos p}{2q^{1/2}(1+q^{1/2}\cos p)} & \frac{q^{1/2}\sin p}{(1+q^{1/2}\cos p)} \end{pmatrix}$$

Upon multiplying the two matrices on the right hand side, we obtain the desired result

$$M^T J M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$$

(b) Show that the function that generates this transformation is

$$F_3 = -(e^Q - 1)^2 \tan p.$$

Soln: Given the fact that $F_3 = F_3(p, Q)$, we must first express the q and P in terms of p and Q, as shown below

$$Q = \log(1 + q^{1/2})\cos p$$

$$\implies q = (e^Q - 1)^2 \sec^2 p$$

$$\implies P = 2(1 + q^{1/2}\cos p)q^{1/2}\sin p = 2\left\{1 + \left(e^Q - 1\right)\sec p\cos p\right\}(e^Q - 1)\sec p \sin p$$

$$\implies P = 2e^Q(e^Q - 1)\tan p.$$

Using the first generating equation, we have

$$\frac{\partial F_3}{\partial p} = -q$$

$$\implies \frac{\partial F_3}{\partial p} = -(e^Q - 1)^2 \sec^2 p$$

$$F_3 = -(e^Q - 1)^2 \tan p + f(Q),$$

where $f(Q)$ is only a function of Q. We substitute this in the second generating equation

$$\begin{aligned}
\frac{\partial F_3}{\partial Q} &= -P \\
\implies -2(e^Q-1)e^Q \tan p + \frac{df}{dQ} &= -2e^Q(e^Q-1)\tan p \\
\implies \frac{df}{dQ} &= 0 \implies f(Q) = \text{constant, which can be ignored}
\end{aligned}$$

Thus

$$F_3(p,Q) = -(e^Q-1)^2 \tan p.$$

6. Prove directly that the transformation

$$\begin{aligned}
Q_1 &= q_1, & P_1 &= p_1 - 2p_2, \\
Q_2 &= p_2, & P_2 &= -2q_1 - q_2
\end{aligned}$$

is canonical and find a generating function.
Soln: We will check the symplectic conditions using the order of variables

$$\eta = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}$$
$$\zeta = \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix},$$

with this

$$\begin{aligned}
M &= \begin{pmatrix} \frac{\partial Q_1}{\partial q_1} & \frac{\partial Q_1}{\partial q_2} & \frac{\partial Q_1}{\partial p_1} & \frac{\partial Q_1}{\partial p_2} \\ \frac{\partial Q_2}{\partial q_1} & \frac{\partial Q_2}{\partial q_2} & \frac{\partial Q_2}{\partial p_1} & \frac{\partial Q_2}{\partial p_2} \\ \frac{\partial P_1}{\partial q_1} & \frac{\partial P_1}{\partial q_2} & \frac{\partial P_1}{\partial p_1} & \frac{\partial P_1}{\partial p_2} \\ \frac{\partial P_2}{\partial q_1} & \frac{\partial P_2}{\partial q_2} & \frac{\partial P_2}{\partial p_1} & \frac{\partial P_2}{\partial p_2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix}
\end{aligned}$$

so that

$$M^T J M = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$
$$= J.$$

Thus, the symplectic condition is satisfied, making the transformation canonical. In order to obtain the generating function, given the structure of transformation equations, it is best to choose a function $F' = F'(p_1, p_2, Q_1, P_2, t)$. Note this generating function does not belong to one of the four standard types. The total generating function in this case will be $F = F'(p_1, p_2, Q_1, P_2, t) - Q_2 P_2 + q_1 p_1 + q_2 p_2$. Now, the condition for canonical transformation is, as usual

$$\begin{aligned} p_1 \dot{q}_1 + p_2 \dot{q}_2 - H &= P_1 \dot{Q}_1 + P_2 \dot{Q}_2 - K + \frac{dF}{dt} \\ &= P_1 \dot{Q}_1 + P_2 \dot{Q}_2 - K - P_2 \dot{Q}_2 - Q_2 \dot{P}_2 \\ &+ p_1 \dot{q}_1 + q_1 \dot{p}_1 + p_2 \dot{q}_2 + q_2 \dot{p}_2 \\ &+ \frac{\partial F'}{\partial p_1} \dot{p}_1 + \frac{\partial F'}{\partial p_2} \dot{p}_2 + \frac{\partial F'}{\partial Q_1} \dot{Q}_1 + \frac{\partial F'}{\partial P_2} \dot{P}_2 + \frac{\partial F'}{\partial t} \end{aligned}$$

which simplifies to

$$\left(\frac{\partial F'}{\partial p_1} + q_1\right) \dot{p}_1 + \left(\frac{\partial F'}{\partial p_2} + q_2\right) \dot{p}_2 + \left(\frac{\partial F'}{\partial Q_1} + P_1\right) \dot{Q}_1 + \left(\frac{\partial F'}{\partial P_2} - Q_2\right) \dot{P}_2 + \left(H + \frac{\partial F'}{\partial t} - K\right) = 0,$$

leading to equations

$$q_1 = -\frac{\partial F'}{\partial p_1} \tag{1}$$
$$q_2 = -\frac{\partial F'}{\partial p_2} \tag{2}$$
$$P_1 = -\frac{\partial F'}{\partial Q_1} \tag{3}$$
$$Q_2 = \frac{\partial F'}{\partial P_2} \tag{4}$$
$$K = H + \frac{\partial F'}{\partial t}. \tag{5}$$

We have to cast the canonical transformation equations such that we can easily integrate the generating function equations. The desired equations are

$$q_1 = Q_1 \tag{6}$$
$$Q_2 = p_2 \tag{7}$$
$$q_2 = -2Q_1 - P_2 \tag{8}$$
$$P_1 = p_1 - 2p_2 \tag{9}$$

Using Eqs. (1) and (6)

$$\begin{aligned} q_1 &= -\frac{\partial F'}{\partial p_1} = Q_1 \\ \implies F' &= -Q_1 p_1 + f(Q_1, p_2, P_2). \end{aligned}$$

Using this in Eqs (2) and (8)

$$\begin{aligned} \frac{\partial f}{\partial p_2} &= 2Q_1 + P_2 \\ \implies f &= 2Q_1 p_2 + P_2 p_2 + g(Q_1, P_2) \\ \implies F' &= -Q_1 p_1 + 2Q_1 p_2 + P_2 p_2 + g(Q_1, P_2) \end{aligned}$$

Using this with Eqs. (3) and (9)

$$\begin{aligned} -p_1 + 2p_2 + \frac{\partial g}{\partial Q_1} &= -p_1 + 2p_2 \\ \implies \frac{\partial g}{\partial Q_1} &= 0 \\ \implies g &= h(P_2) \\ \implies F' &= -Q_1 p_1 + 2Q_1 p_2 + P_2 p_2 + h(P_2). \end{aligned}$$

Using this in Eqs. (4) and (7), we have

$$\begin{aligned} p_2 + \frac{dh}{dP_2} &= p_2 \\ \implies \frac{dh}{dP_2} &= 0 \implies h = 0 (\text{by choice}), \end{aligned}$$

leading to the final expression for the generating function

$$F' = -Q_1 p_1 + 2Q_1 p_2 + P_2 p_2.$$

7. (a) Using the fundamental Poisson brackets find the values of α and β for which the equations

$$Q = q^\alpha \cos \beta p, \qquad P = q^\alpha \sin \beta p$$

represent a canonical transformation.
Soln: The fundamental Poisson brackets should remain invariant under a canonical transformation, i.e.,

$$\begin{aligned} [Q,P]_{q,p} &= 1 \\ \implies \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} &= 1 \\ \implies (\alpha q^{\alpha-1}\cos\beta p)(\beta q^{\alpha}\cos\beta p) - (-\beta q^{\alpha}\sin\beta p)(\alpha q^{\alpha-1}\sin\beta p) &= 1 \\ \implies \alpha\beta q^{2\alpha-1}(\sin^2\beta p + \cos^2\beta p) &= 1 \\ \alpha\beta q^{2\alpha-1} &= 1. \end{aligned}$$

This equation is satisfied if $2\alpha - 1 = 0 \implies \alpha = 1/2$ and $\beta = 1/\alpha = 2$.

(b) For what values of α and β do these equations represent an extended canonical transformation? Find a generating function of the F_3 form for the transformation.
Soln: When $\alpha = 1/2$ and β is taken to be an arbitrary constant, we have

$$[Q,P]_{q,p} = \frac{\beta}{2},$$

which represents an extended canonical transformation for any value of $\beta \neq 2$. Now, our transformation equations are

$$\begin{aligned} Q &= q^{1/2}\cos\beta p \\ P &= q^{1/2}\sin\beta p \end{aligned}$$

For extended canonical transformation for a system with one degree of freedom, we have

$$\lambda(p\dot{q} - H) = P\dot{Q} - K + \frac{dF}{dt}.$$

When $F = F_3(p,Q,t) + \lambda pq$, we obtain

$$\lambda(p\dot{q} - H) = P\dot{Q} - K + \lambda\dot{p}q + \lambda p\dot{q} + \frac{\partial F_3}{\partial p}\dot{p} + \frac{\partial F_3}{\partial Q}\dot{Q} + \frac{\partial F_3}{\partial t},$$

which leads to

$$\frac{\partial F_3}{\partial p} = \lambda q \tag{10}$$

$$\frac{\partial F_3}{\partial Q} = -P \tag{11}$$

$$K = \lambda H + \frac{\partial F_3}{\partial t} \tag{12}$$

We first express P and q in terms of p and Q, as below

$$q = Q^2\sec^2\beta p \tag{13}$$

$$P = Q\sec\beta p\sin\beta p = Q\tan\beta p \tag{14}$$

Combining Eqs. (10) and (13), we have

$$\frac{\partial F_3}{\partial p} = \lambda Q^2 \sec^2 \beta p$$
$$\implies F_3 = \frac{\lambda}{\beta} Q^2 \tan \beta p + f(Q)$$

Using this in combination with Eqs. (11) and (14), we have

$$2\frac{\lambda}{\beta} Q \tan \beta p + \frac{df}{dQ} = Q \tan \beta p.$$
$$\implies \frac{df}{dQ} = (1 - \frac{2\lambda}{\beta}) Q \tan \beta p$$
$$\implies f(Q) = \frac{1}{2}(1 - \frac{2\lambda}{\beta}) Q^2 \tan \beta p,$$

leading to the final expression for the generating function

$$F_3 = \frac{1}{2} Q^2 \tan \beta p$$

8. Show by the use of Poisson brackets that for a one-dimensional harmonic oscillator, there is a constant of motion u defined as

$$u(q, p, t) = \ln(p + im\omega q) - i\omega t, \qquad \omega = \sqrt{\frac{k}{m}}.$$

Soln: We know that a quantity u is a constant of motion provided

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} = 0.$$

For the 1D simple harmonic oscillator, the Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2,$$

so that

$$\begin{aligned}
[u, H] + \frac{\partial u}{\partial t} &= \left(\frac{\partial u}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial H}{\partial q} \right) - i\omega \\
&= \left(\frac{im\omega}{p + im\omega q} \right) (\frac{p}{m}) - \left(\frac{1}{p + im\omega q} \right) m\omega^2 q - i\omega \\
&= \frac{i\omega p - m\omega^2 q}{p + im\omega q} - i\omega \\
&= \frac{i\omega p - m\omega^2 q - i\omega p + m\omega^2 q}{p + im\omega q} \\
&= 0,
\end{aligned}$$

hence, u is a constant of motion.

9. A system of two degrees of freedom is described by the Hamiltonian

$$H = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2,$$

where a and b are constants. Show that

$$F_1 = \frac{p_1 - a q_1}{q_2} \quad \text{and} \quad F_2 = q_1 q_2$$

are constants of the motion.
Soln: Because both F_1 and F_2 have no explicit dependence on time $(\frac{\partial F_1}{\partial t} = \frac{\partial F_2}{\partial t} = 0)$, therefore, using the general result above, they will be constants of motion, if their Poisson brackets with the Hamiltonian vanish, i.e.,

$$\frac{dF_1}{dt} = [F_1, H] = 0$$
$$\frac{dF_2}{dt} = [F_2, H] = 0.$$

Let us calculate these Poisson brackets

$$\begin{aligned}
[F_1, H] &= \sum_{i=1}^{2} \left(\frac{\partial F_1}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F_1}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \\
&= \left\{ (-a/q_2) q_1 + (-\frac{p_1 - a q_1}{q_2^2})(-q_2) - (\frac{1}{q_2})(p_1 - 2a q_1) - 0 \right\} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
[F_2, H] &= \sum_{i=1}^{2} \left(\frac{\partial F_2}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F_2}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \\
&= \{ q_2 q_1 + q_1(-q_2) - 0 - 0 \} \\
&= 0.
\end{aligned}$$

Thus both F_1 and F_2 are constants of motion.