PH403: Quantum Mechanics I
Tutorial Sheet 2

This tutorial sheet deals with solutions of time-independent Schrödinger equation for simple potentials.

1. Consider the time-independent Schrödinger equation for a particle confined in one dimension, and exposed to a potential $V(x)$. Show that across a finite discontinuity in the potential, the first derivative of the wave function $\psi(x)$, i.e., $\psi'(x)$, is continuous.

2. Consider a particle in a 1D box, i.e., a particle exposed to a potential $V(x)$ which satisfies
$$V(x) = \begin{cases} \infty \text{ for } x < 0 \text{ and } x > a \\ 0 \text{ otherwise.} \end{cases}$$
Solve the time-independent Schrödinger equation for this system to obtain its eigenvalues and eigenfunctions. How will you generalize this solution if this potential well were in higher dimensions, i.e., 2D, and 3D.

3. Bound states in a finite potential well. Consider a particle of of mass m and energy E, subject to a potential defined by
$$V(x) = \begin{cases} -V_0 \text{ for } x \geq 0 \text{ and } x \leq a \\ 0 \text{ otherwise,} \end{cases}$$
where $V_0 > 0$. Obtain the solutions of the time-independent Schrödinger equation for this system if $0 > E > -V_0$.

4. Consider a particle of mass m and energy E, incident from the left on a potential barrier defined by
$$V(x) = \begin{cases} 0 \text{ for } x < 0 \\ V_0 \text{ x>0,} \end{cases}$$
where $V_0 > 0$. Solve the Schrödinger equation for this system when (a) $E < V_0$, and (b) $E > V_0$. For each case, calculate the reflection coefficient R, and transmission coefficient T. Check any book on quantum mechanics to see the definitions of R and T.

5. Scattering from a finite 1D potential barrier. Consider a particle of mass m and energy E, incident from the left on a potential barrier defined by
$$V(x) = \begin{cases} 0 \text{ for } x < 0 \text{ and } x > a \\ V_0 \text{ otherwise,} \end{cases}$$
where $V_0 > 0$. Solve the Schrödinger equation for this system when (a) $E < V_0$, and (b) $E > V_0$. For each case, calculate the reflection coefficient R, and transmission coefficient T. Note that as far as classical mechanics is concerned, no transmission is possible when $E < V_0$. Nonzero value of T obtained quantum mechanically for this case is a manifestation of a phenomenon called tunneling. The fact that tunneling has been observed under a variety of situations experimentally, is yet another feather in the cap of quantum mechanics.

6. Bound state of particle in a "delta-function" potential. Consider a particle which is exposed to the 1D potential

$$V(x) = -\alpha\delta(x),$$

where $\alpha > 0$ is a constant with suitable dimensions.

   (a) Integrate the time-independent Schrödinger equation between $-\epsilon$ and ϵ to obtain how the wave function and its first derivative behave at the discontinuity at $x = 0$.

   (b) Assuming that the energy of the particle is $E < 0$, and the wave function is bound, obtain the possible values of E, and the corresponding wave functions.

   (c) Plot the wave functions and estimate their width Δx.

   (d) What is the probability $d\mathcal{P}(p)$ that a measurement of the momentum of the particle in one of the normalized stationary states calculated above will give a result included between p and $p + dp$. For what value of p is this probability maximum?

7. Consider the potential of the previous problem. Assume that a particle with energy $E > 0$ is incident from the left on this potential. Calculate the coefficients R and T for this system in terms of the dimensionless parameter E/E_L, where $E_L = -m\alpha^2/2\hbar^2$. Study and analyze the variations of R and T with respect to E.