EP 307: Quantum Mechanics I
Tutorial Sheet 4

Problems in this tutorial sheet deal with the mathematical methods of quantum mechanics as discussed in chapter 2 of the book by Cohen-Tannoudji *et al.* Read complement B of chapter II to revise some important concepts related to linear operators, their functions, and commutators.

1. $|\phi_n\rangle$ are the eigenstates of a Hermitian operator H, which could be the Hamiltonian of a physical system. Assume that the states $|\phi_n\rangle$ form a discrete orthonormal basis. The operator $U(m,n)$ is defined by

$$U(m,n) = |\phi_m\rangle\langle\phi_n|.$$

   (a) Calculate the Hermitian conjugate $U^\dagger(m,n)$, of $U(m,n)$

   (b) Calculate the commutator $[H, U(m,n)]$

   (c) Prove that $U(m,n)U^\dagger(p,q) = \delta_{nq}U(m,p)$

   (d) Calculate $\mathrm{Tr}(U(m,n))$

   (e) Let A be an operator, with matrix elements $A_{mn} = \langle\phi_m|A|\phi_n\rangle$. Prove that $A = \sum_{m,n} A_{mn}U(m,n)$

   (f) Show that $A_{pq} = \mathrm{Tr}(AU^\dagger(p,q))$

2. In a two-dimensional vector space, consider the operator whose matrix, in an orthonormal basis $\{|1\rangle, |2\rangle\}$, is written as

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

   (a) Is σ_y Hermitian? Calculate its eigenvalues and eigenvectors (giving their normalized expansion in terms of the $\{|1\rangle, |2\rangle\}$ basis.

   (b) Calculate the matrices which represent projectors on to these eigenvectors. Then verify that they satisfy the orthogonality and closure relations.

3. Let K be the operator defined by $K = |\phi\rangle\langle\psi|$, where $|\phi\rangle$, $|\psi\rangle \in \mathcal{E}$.

   (a) Under what conditions is K Hermitian?

   (b) Calculate K^2. Under what conditions is K a projector?

   (c) Show that K can always be written in the form $K = \lambda P_1 P_2$, where λ is a constant to be calculated and P_1 and P_2 are projectors.

4. The σ_x matrix is defined by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Prove the relation

$$e^{i\alpha\sigma_x} = I\cos\alpha + i\sigma_x\sin\alpha,$$

   where I is the 2×2 identity matrix and α is a number.

5. Prove the following properties of commutators

   (a) $[A, B] = -[B, A]$
   (b) $[A, B]^\dagger = [B^\dagger, A^\dagger]$
   (c) $[A, B + C] = [A, B] + [A, C]$
   (d) $[A, BC] = [A, B]C + B[A, C]$
   (e) $[AB, C] = A[B, C] + [A, C]B$
   (f) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

6. Consider the Hamiltonian H of a particle in a one-dimensional problem defined by

$$H = \frac{P^2}{2m} + V(X),$$

   where X and P are the position and momentum operators which satisfy $[X, P] = i\hbar$. Assume that H has a discrete spectrum described by the eigenvalue equation $H|\phi_n\rangle = E_n|\phi_n\rangle$.

   (a) by evaluating $\langle\phi_n|[X, H]|\phi_{n'}\rangle$, prove that

$$\langle\phi_n|P|\phi_{n'}\rangle = \alpha\langle\phi_n|X|\phi_{n'}\rangle,$$

   and determine the value of α

   (b) From this, deduce, using the closure relation, the equation

$$\sum_{n'}(E_n - E_{n'})^2|\langle\phi_n|X|\phi_{n'}\rangle|^2 = \frac{\hbar^2}{m^2}\langle\phi_n|P^2|\phi_n\rangle$$

7. Let H be the Hamiltonian of a physical system with eigenspectrum defined by $H|\phi_n\rangle = E_n|\phi_n\rangle$.

   (a) For an arbitrary operator A prove that $\langle\phi_n|[A, H]|\phi_n\rangle = 0$.
   (b) If the Hamiltonian in question represents a particle of mass m confined along the x direction with

$$H = \frac{P^2}{2m} + V(X),$$

   i. In terms of P, X, and $V(X)$, compute the commutators: $[H, P]$, $[H, X]$, and $[H, XP]$.
   ii. Show that $\langle\phi_n|P|\phi_n\rangle = 0$.
   iii. Establish a relation between $K = \langle\phi_n|\frac{P^2}{2m}|\phi_n\rangle$ and $\langle\phi_n|X\frac{dV}{dX}|\phi_n\rangle$. How is K related to $\langle\phi_n|V(X)|\phi_n\rangle$, when $V(X) = V_0X^\lambda$ ($\lambda = 2, 4, 6, \ldots; V_0 > 0$).

8. Using the relation $\langle x|p\rangle = (2\pi\hbar)^{-1/2}e^{ipx/\hbar}$, find the expression $\langle x|XP|\psi\rangle$ and $\langle x|PX|\psi\rangle$ in terms of $\psi(x)$. Can these results be found directly by using the fact that in the $|x\rangle$ representation, P acts like $\frac{\hbar}{i}\frac{d}{dx}$?

9. Consider a three-dimensional state space spanned by the orthonormal basis kets $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$. Consider two operators L_z and S defined by

$$\begin{array}{lll} L_z|u_1\rangle = |u_1\rangle & L_z|u_2\rangle = 0 & L_z|u_3\rangle = -|u_3\rangle \\ S|u_1\rangle = |u_3\rangle & S|u_2\rangle = |u_2\rangle & S|u_3\rangle = |u_1\rangle, \end{array}$$

(a) Obtain the matrix representations of operators L_z, L_z^2, S, and S^2 with respect to this basis. Are these operators observables?

(b) Give the form of the most general matrix which represents an operator which commutes with L_z. Same question for L_z^2 and S^2.

(c) Do L_z^2 and S form a C.S.C.O.? Give a basis of common eigenvectors.