PH403: Quantum Mechanics I
Tutorial Sheet 5

Problems in this tutorial sheet deal with the postulates of quantum mechanics as discussed in chapter 3 of the book by Cohen-Tannoudji *et al.*

1. In a one-dimensional problem, consider a particle whose wave function is

$$\psi(x) = N \frac{e^{ip_0 x/\hbar}}{\sqrt{x^2 + a^2}},$$

where a and p_0 are real constants and N is a normalization coefficient.

   (a) Determine N so that $\psi(x)$ is normalized.

   (b) The position of the particle is measured. What is the probability of finding a result between $-\frac{a}{\sqrt{3}}$ and $\frac{a}{\sqrt{3}}$?

   (c) Calculate $\langle p \rangle$ the expectation value of the momentum of the particle corresponding to $\psi(x)$.

2. Consider, in a one-dimensional problem, a particle of mass m whose wave function at time t is $\psi(x, t)$.

   (a) At time t, the distance d of this particle from the origin is measured. Write, as a function of $\psi(x, t)$, the probability $P(d_0)$ of finding a result greater than a given length d_0. What are the limits of $P(d_0)$ when $d_0 \to 0$ and $d_0 \to \infty$?

   (b) Instead of performing the measurement of question a, one measures the velocity v of the particle at time t. Express, as a function of $\psi(x, t)$, the probability of finding a result greater than a given value v_0.

3. Consider the three-dimensional wave function

$$\psi(x, y, z) = N e^{-\{\frac{|x|}{2a} + \frac{|y|}{2b} + \frac{|z|}{2c}\}}$$

where a, b, and c are three positive lengths.

   (a) Calculate the constant N which normalizes ψ.

   (b) Calculate the probability that a measurement of X will yield a result included between 0 and a.

   (c) Calculate the probability that a simultaneous measurement of Y and Z will yield results included between $-b$ and b, and $-c$ and c.

   (d) Calculate the probability that a measurement of the momentum will yield a result included in the element $dp_x dp_y dp_z$ centered at the point $p_x = p_y = 0$; $p_z = \hbar/c$.

4. Consider two canonically conjugate operators Q and P, which satisfy the commutation relation $[Q, P] = i\hbar$. Our aim here is to prove that $\Delta P \Delta Q \geq \hbar/2$ by using the following sequence of arguments

(a) By using the fact that the norm of the ket $|\phi\rangle = (Q + i\lambda P)|\psi\rangle$ is non-negative ($|\psi\rangle$ is an arbitrary ket) argue that $\langle P^2\rangle\langle Q^2\rangle \geq \frac{\hbar^2}{4}$

(b) Define $P' = P - \langle P\rangle$ and $Q' = Q - \langle Q\rangle$ and compute the commutator $[P', Q']$.

(c) Then argue from the results of part (a) that $\langle P'^2\rangle\langle Q'^2\rangle \geq \frac{\hbar^2}{4}$ leading to the result in question.

5. In a one-dimensional problem, consider a particle of potential energy $V(X) = -fX$, where f is a positive constant.

   (a) Write Ehrenfest's theorem for the mean values of the position X and the momentum P of the particle. Integrate these equations; compare with the classical motion.

   (b) Show that the root-mean-square deviation ΔP does not vary over time.

   (c) Write the Schrödinger equation in the $\{|p\rangle\}$ representation. Deduce from it a relation between $\frac{\partial}{\partial t}|\langle p|\psi(t)\rangle|^2$ and $\frac{\partial}{\partial p}|\langle p|\psi(t)\rangle|^2$. Integrate the equation thus obtained; give a physical interpretation.

6. Earlier we solved the problem of a particle in a box defined by the potential $V(x)$ which satisfies

$$V(x) = \begin{cases} \infty \text{ for } x < 0 \text{ and } x > a \\ 0 \text{ otherwise.} \end{cases}$$

   Assume that eigenstates of the Hamiltonian H are $|\phi_n\rangle$ with eigenvalues $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$. The state of the particle at the instant $t = 0$ is

$$|\psi(0)\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle + a_3|\phi_3\rangle + a_4|\phi_4\rangle.$$

   (a) What is the probability, when the energy of the particle in the state $|\psi(0)\rangle$ is measured, of finding a value smaller than $\frac{3\pi^2\hbar^2}{ma^2}$?

   (b) What is the expectation value and what is the root-mean-square deviation of the energy of the particle in the state $|\psi(0)\rangle$?

   (c) Calculate the state vector $|\psi(t)\rangle$ at the instant t. Do the results found in (a) and (b) at the instant $t = 0$ remain valid at an arbitrary time t?

   (d) When the energy is measured, the result $\frac{8\pi^2\hbar^2}{ma^2}$ is found. After the measurement, what is the state of the system? What is the result if the energy is measured again?

7. Consider a physical system whose state space, which is three-dimensional, is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$. In this basis, the Hamiltonian operator H of the system and the two observables A and B are written

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ω_0, a, and b are positive real constants. At $t = 0$, the system is described by the state ket

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle.$$

(a) At time $t = 0$, the energy of the system is measured. What values can be found, and with what probabilities? Calculate for the system in $|\psi(0)\rangle$, the mean value $\langle H\rangle$ and the root-mean square deviation ΔH.

(b) Instead of measuring H at time $t = 0$, one measures A; what results can be found, and with what probabilities? What is the state vector immediately after the measurement?

(c) Calculate the state vector $|\psi(t)\rangle$ of the system at time t?

(d) Calculate the expectation values $\langle A\rangle(t)$ and $\langle B\rangle(t)$ of A and B at time t. What comments can be made?

(e) What results are obtained if the observable A is measured at time t? Same question for B. Interpret your results.