PH403: Quantum Mechanics I
Tutorial Sheet 6

Problems in this tutorial sheet deal with the Stern-Gerlach experiment and the related problems as discussed in chapter 4 of the book by Cohen-Tannoudji *et al.*

1. Consider a spin-1/2 particle of magnetic moment $\mathbf{M} = \gamma \mathbf{S}$. The spin state space is spanned by the basis of the $|+\rangle$ and $|-\rangle$, eigenvectors of S_z with eigenvalues $\hbar/2$ and $-\hbar/2$. At the time $t = 0$, the state of the system is $|\psi(0)\rangle = |+\rangle$.

   (a) If the observable S_x is measured at time $t = 0$, what results can be found and with what probabilities?

   (b) Instead of performing the preceding measurement, we let the system evolve under the influence of a magnetic field of magnitude B_0, parallel to the $y-$axis. Calculate in the $\{|+\rangle,\ |-\rangle\}$ basis the state of the system at time t.

   (c) At this time t, we measure the observables S_x, S_y, and S_z. What values can we find, and with what probabilities? What relation must exist between B_0 and t for the final result of one of the measurements to be certain? Give a physical interpretation of this condition.

2. Consider a spin-1/2 particle, as in the previous exercise.

   (a) At time $t = 0$, we measure S_y and find $+\hbar/2$. What is the state vector $|\psi(0)\rangle$ immediately after the measurement?

   (b) Immediately after this measurement, we apply a uniform time-dependent field along the z-axis. The spin Hamiltonian $H(t)$ is then written as

   $$H(t) = \omega_0(t) S_z$$

   Assume that $\omega_0(t)$ is zero for $t < 0$ and $t > T$, and linearly increases from 0 to ω_0 when $0 \leq t \leq T$ (T is a given parameter whose dimensions are those of time). Show that at time t the state vector can be written

   $$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\{e^{i\theta(t)}|+\rangle + ie^{-i\theta(t)}|-\rangle\},$$

   where $\theta(t)$ is a real function of t which you have to compute.

   (c) At a time $t = \tau > T$, we measure S_y. What results can we find, and with what probabilities? Determine the relation which must exist between ω_0 and T in order for us to be sure of the result. Give a physical interpretation.

3. Consider a spin-1/2 particle placed in a magnetic field $\mathbf{B}_0 = \frac{1}{\sqrt{2}}(B_0, 0, B_0)$. Notation is the same as in exercise 1.

   (a) Calculate the matrix representing, in the $\{|+\rangle,\ |-\rangle\}$ basis, the Hamiltonian operator H, of the system.

(b) Calculate the eigenvalues and eigenvectors of H.

(c) The system at time $t = 0$ is in the state $|-\rangle$. What values can be found if the energy is measured, and with what probabilities?

(d) Calculate the state vector $|\psi(t)\rangle$ at time t. At this instant, S_x is measured: what is the mean value of the results that can be obtained? Give a geometrical interpretation.

4. Consider a spin-1/2, of magnetic moment $\mathbf{M} = \gamma\mathbf{S}$, placed in a magnetic field $\mathbf{B}_0$ of components $B_x = -\omega_x/\gamma$, $B_y = -\omega_y/\gamma$, $B_z = -\omega_z/\gamma$. We set $\omega_0 = -\gamma|\mathbf{B}_0|$.

   (a) Show that the time evolution operator of this spin is

   $$U(t,0) = e^{-iMt}$$

   where M is given by

   $$M = \frac{1}{\hbar}\{\omega_x S_x + \omega_y S_y + \omega_z S_z\}$$

   Calculate the matrix representation of M in the $\{|+\rangle,\ |-\rangle\}$ basis and show that

   $$M^2 = \frac{1}{4}(\omega_x^2 + \omega_y^2 + \omega_z^2) = (\frac{\omega_0}{2})^2$$

   (b) Put the time evolution operator into the form

   $$U(t,0) = \cos\left(\frac{\omega_0 t}{2}\right) - \frac{2i}{\omega_0} M \sin\left(\frac{\omega_0 t}{2}\right)$$

   (c) Consider a spin which at time $t = 0$ is in the state $|\psi(0)\rangle = |+\rangle$. Show that the probability $P_{++}(t)$ of finding it in state $|+\rangle$ at time t is

   $$P_{++}(t) = |\langle +|U(t,0)|+\rangle|^2$$

   and derive the relation

   $$P_{++}(t) = 1 - \frac{\omega_x^2 + \omega_y^2}{\omega_0^2}\sin^2\left(\frac{\omega_0 t}{2}\right)$$

5. Consider a two-level system with orthonormal basis vectors $\{|\phi_1\rangle, |\phi_2\rangle\}$ and the Hamiltonian $H = H_0 + W$ defined by

   $$H_0|\phi_i\rangle = E_i|\phi_i\rangle \quad (i = 1, 2)$$

   and

   $$W|\phi_1\rangle = W_{21}|\phi_2\rangle; \quad W|\phi_2\rangle = W_{12}|\phi_1\rangle$$

   where W_{ji} 's are the matrix elements of the operator W, and $W_{12} = W_{21}^*$.

   (a) Construct the matrix representing the Hamiltonian H in the given basis.

   (b) Obtain the eigenvalues and eigenvectors of H.

   (c) Consider a ket which at time $t = 0$ is given by $|\psi(0)\rangle = |\phi_1\rangle$. Compute the expression for the probability $P_{12}(t)$ that at a later time t, the system will be in state $|\phi_2\rangle$. This probability is defined by $P_{12}(t) = |\langle\phi_2|\psi(t)\rangle|^2$. Plot $P_{12}(t)$ as a function of time t.