PH403: Quantum Mechanics I
Tutorial Sheet 7

Problems in this tutorial sheet deal with the quantum mechanics of a simple-harmonic oscillator as discussed in chapter 5 of the book by Cohen-Tannoudji *et al.*

1. Obtain and integrate the equations of motion for the expectation values of the X and P operators for a simple-harmonic oscillator of mass m and frequency ω.

2. Consider a harmonic oscillator of mass m and angular frequency ω. At time $t = 0$, the state of this oscillator is given by

$$|\psi(0)\rangle = \sum_n c_n |\phi_n\rangle,$$

where $|\phi_n\rangle$ are stationary states with energies $(n + 1/2)\hbar\omega$.

(a) What is the probability P that a measurement of the oscillator's energy performed at an arbitrary time $t > 0$, will yield a result greater than $2\hbar\omega$? When $P = 0$, what are the nonzero coefficients c_n?

(b) From now on assume that only c_0 and c_1 are different from zero. Write the normalization coefficient for $|\psi(0)\rangle$ and the mean value $\langle H\rangle$ of the energy in terms of c_0 and c_1. With the additional requirement $\langle H\rangle = \hbar\omega$, calculate $|c_0|^2$and $|c_1|^2$.

(c) As the normalized state vector $|\psi(0)\rangle$ is defined only to within a global phase factor, we fix this factor by choosing c_0 to be real and positive. We set $c_1 = |c_1|e^{i\theta}$. We assume that $\langle H\rangle = \hbar\omega$ and that

$$\langle X\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}.$$

Calculate θ.

3. Two particles of the same mass m, with positions X_1 and X_2 and momenta P_1 and P_2, are subject to the same potential

$$V(X) = \frac{1}{2}m\omega^2 X^2.$$

The two particles do not interact with each other.

(a) Show that the Hamiltonian H of the two-particle system can be written as

$$H = H_1 + H_2,$$

where H_1/H_2 act on the state space of particle 1/2.

(b) Calculate the energies of the two-particle system, their degree of degeneracy, and the corresponding wave functions.

(c) Do H, H_1, and H_2 form a C.S.C.O? We denote by $|\Phi_{n1,n2}\rangle$ the eigenvectors common to H_1 and H_2. Write the orthonormalization and closure relation for the states $|\Phi_{n1,n2}\rangle$.

(d) Consider a system which, at $t = 0$, is in the state

$$|\psi(0)\rangle = \frac{1}{2}(|\Phi_{0,0}\rangle + |\Phi_{1,0}\rangle + |\Phi_{0,1}\rangle + |\Phi_{1,1}\rangle).$$

What results can be obtained, and with what probabilities, if at this time one measures: (i) the total energy of the system, and (ii) energy of particle 1?

4. This is a continuation of the previous problem and, therefore, uses the same notations. Assume that at $t = 0$, the system is in state $|\psi(0)\rangle$ given in the previous exercise. At $t = 0$, one measures the total energy H and finds the result $2\hbar\omega$.

   (a) Calculate the mean values (expectation values) of the position, the momentum, and the energy of particles 1 and 2 at an arbitrary $t > 0$.

   (b) At $t > 0$, one measures the energy of particle 1. What results can be found, and with what probabilities? Same question for a measurement of the position of particle 1. Plot the curve for the corresponding probability density.

5. We know that the Hamiltonian of a simple-harmonic oscillator can be written as $H = \hbar\omega(a^\dagger a + \frac{1}{2})$, and its time evolution operator can be written as $U(t, 0) = e^{-iHt}$.

   (a) Consider the operators

$$\begin{aligned} \tilde{a}(t) &= U^\dagger(t, 0)aU(t, 0) \\ \tilde{a}^\dagger(t) &= U^\dagger(t, 0)a^\dagger U(t, 0) \end{aligned}$$

   By calculating their actions on the kets $|\phi_n\rangle$, find the expressions for $\tilde{a}(t)$ and $\tilde{a}^\dagger(t)$ in terms of a and $a^\dagger$.

   (b) Calculate the operators $\tilde{X}(t)$ and $\tilde{P}(t)$ defined by

$$\begin{aligned} \tilde{X}(t) &= U^\dagger(t, 0)XU(t, 0) \\ \tilde{P}(t) &= U^\dagger(t, 0)PU(t, 0). \end{aligned}$$

   How can the relations so obtained be interpreted?

   (c) Show that $U^\dagger\left(\frac{\pi}{2\omega}, 0\right)|x\rangle$ is an eigenvector of P and specify its eigenvalue. Similarly, establish that $U^\dagger\left(\frac{\pi}{2\omega}, 0\right)|p\rangle$ is an eigenvector of X.

6. Consider the operator $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$

   (a) Prove that

$$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a}.$$

   Hint: you should use Glauber's Formula

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]},$$

   which is valid when both the operators A and B commute with their commutator $[A, B]$.

(b) Prove that

$$D(\alpha)|0\rangle = |\alpha\rangle,$$

where $|0\rangle$ represents the harmonic oscillator eigenstate with $n = 0$, and $|\alpha\rangle$ represents the coherent state.

(c) Prove that

$$|\langle\alpha|\alpha'\rangle|^2 = e^{-|\alpha-\alpha'|^2}.$$

This shows that coherent states form a non-orthogonal set of vectors.