PH 422: Quantum Mechanics II
Tutorial Sheet 1: Solution

This tutorial sheet contains problems related to the addition of angular momenta for quantum mechanical particles.

1. Verify the values of the following C-G coefficients

   (a) $\langle j1j0|j1jj\rangle = \sqrt{\frac{j}{j+1}}$
   Soln: In this CGC, $j_1 = j$, $j_2 = 1$,$m_1 = j$, $m_2 = 0$, $j = j$ and $m = j$, which makes it of the form $\langle j_1j_2j_1j - j_1|j_1j_2jj\rangle$. CGCs of this form can be computed by using the normalization condition, coupled with the recursion relations. We use the normalization condition

$$\langle j1jj|j1jj\rangle = 1$$

resolution of identity in the uncoupled respresentation $\sum_{m_1,m_2} |j1m_1m_2\rangle\langle j1m_1m_2| = 1$

using this in the first equation, we obtain $\sum_{m_1,m_2} \langle j1m_1m_2|j1jj\rangle^2 = 1$

$$\sum_{m_1}\langle j1m_1 - 1|j1jj\rangle^2 + \sum_{m_1}\langle j1m_10|j1jj\rangle^2 + \sum_{m_1}\langle j1m_11|j1jj\rangle^2 = 1$$

Above first term will go to zero because $m = m_1 + m_2 = j$ requires $m_1 = j + 1$. Thus, we have

$$\langle j1j0|j1jj\rangle^2 + \langle j1j - 11|j1jj\rangle^2 = 1. \tag{1}$$

Using recursion relation with the upper sign

$$\begin{aligned}
&\sqrt{(j \mp m)(j \pm m + 1)}\,\langle j_1j_2m_1m_2 \mid j_1j_2jm \pm 1\rangle \\
= &\sqrt{(j_1 \pm m_1)(j_1 \mp m_1 + 1)}\,\langle j_1j_2m_1 \mp 1m_2 \mid j_1j_2jm\rangle \\
+ &\sqrt{(j_2 \pm m_2)\,(j_2 \mp m_2 + 1)}\,\langle j_1j_2m_1m_2 \mp 1 \mid j_1j_2jm\rangle
\end{aligned} \tag{2}$$

and $m = j$, $j_1 = j$, $j_2 = 1$, $m_1 = j$, and $m_2 = 1$, we obtain

$$\sqrt{2j}\langle j1j - 11|j1jj\rangle = -\sqrt{2}\langle j1j0|j1jj\rangle$$

$$\langle j1j - 11|j1jj\rangle = -\frac{1}{\sqrt{j}}\langle j1j0|j1jj\rangle$$

Using this in Eq. 1, we obtain the desired result, with the positive choice for the phase

$$\langle j1j0|j1jj\rangle^2\left(1 + \frac{1}{j}\right) = 1$$

$$\langle j1j0|j1jj\rangle = \sqrt{\frac{j}{j+1}}$$

(b) $\langle j2j0|j2jj\rangle = \sqrt{\frac{j(2j-1)}{(j+1)(2j+3)}}$
Soln: This CGC is also of the form $\langle j_1 j_2 j_1 j - j_1 | j_1 j_2 j j\rangle$, with $j_1 = j$, $j_2 = 2$, $m_1 = j$, $m_2 = 0$, $j = j$, and $m = j$. Again, we start out with the normalization condition, which combined with $m = m_1 + m_2$ conservation, leads to

$$\sum_{m_1,m_2} \langle j2m_1m_2|j2jj\rangle^2 = 1$$

$$\langle j, 2j-2, 2|j2jj\rangle^2 + \langle j, 2, j-1, 1|j2jj\rangle^2 + \langle j, 2, j, 0|j2jj\rangle^2 = 1 \qquad (3)$$

Again, using the recursion relations of Eq. 2 with the upper sign, and $j_1 = j$, $m = j$, $j_2 = 2$, $m_1 = j$, and $m_2 = 1$, we obtain

$$\sqrt{2j}\langle j2j-11|j2jj\rangle = -\sqrt{6}\langle j2j0|j2jj\rangle$$

$$\langle j2j-11|j2jj\rangle = -\sqrt{\frac{3}{j}}\langle j2j0|j2jj\rangle. \qquad (4)$$

On using the same recursion relation with $j_1 = j$, $m = j$, $j_2 = 2$, $m_1 = j-1$, and $m_2 = 2$, we obtain

$$\sqrt{2(2j-1)}\langle j2j-22|j2jj\rangle = -2\langle j2j-11|j2jj\rangle$$

$$\langle j2j-22|j2jj\rangle = -\sqrt{\frac{2}{2j-1}}\langle j2j-11|j2jj\rangle$$

Using Eq. 4 we obtain

$$\langle j2j-22|j2jj\rangle = \sqrt{\frac{6}{j(2j-1)}}\langle j2j0|j2jj\rangle \qquad (5)$$

Using Eqs. 4 and 5, in Eq. 3, we obtain

$$\langle j2j0|j2jj\rangle^2 \left(1 + \frac{3}{j} + \frac{6}{j(2j-1)}\right) = 1$$

$$\Longrightarrow \langle j2j0|j2jj\rangle^2 \left\{\frac{(2j+3)(j+1)}{j(2j-1)}\right\} = 1$$

$$\Longrightarrow \langle j2j0|j2jj\rangle = \sqrt{\frac{j(2j-1)}{(2j+3)(j+1)}}.$$

2. Compute the following C-G coefficients

(a) $\langle j_1, 1/2, m-1/2, 1/2|j_1, 1/2, j_1 \pm 1/2, m\rangle = \pm\sqrt{\frac{j_1 \pm m + 1/2}{2j_1+1}}$
Soln: (i) We first start to evaluate the CGC $\langle j_1, 1/2, m-1/2, 1/2|j_1 1/2, j_1 + 1/2, m\rangle$, using the recursion relation

$$\sqrt{(j+m)(j-m+1)}\langle j_1, j_2, m_1, m_2|j_1, j_2, j, m-1\rangle =$$
$$\sqrt{(j_1-m_1)(j_1+m_1+1)}\langle j_1, j_2, m_1+1, m_2|j_1, j_2, j, m\rangle$$
$$+\sqrt{(j_2-m_2)(j_2+m_2+1)}\langle j_1, j_2, m_1, m_2+1|j_1, j_2, j, m\rangle$$

In this relation when we substitute $j_1 = j_1$, $m_1 = m-\frac{1}{2}$, $j_2 = \frac{1}{2}$,$m_2 = \frac{1}{2}$,$j = j_1+\frac{1}{2}$, and $m \to m+1$, the second term on the RHS of the recursion relation vanishes leading to

$$\sqrt{(j_1+\frac{1}{2}+m+1)(j_1+\frac{1}{2}-m-1+1)}\langle j_1,\frac{1}{2},m-\frac{1}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle$$
$$= \sqrt{(j_1+m-\frac{1}{2}+1)(j_1-m+\frac{1}{2})}\langle j_1,\frac{1}{2},m+\frac{1}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m+1\rangle$$

This implies

$$\langle j_1,\frac{1}{2},m-\frac{1}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle = \sqrt{\frac{(j_1+m+\frac{1}{2})}{(j_1+m+\frac{3}{2})}}\langle j_1,\frac{1}{2},m+\frac{1}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m+1\rangle. \tag{6}$$

We apply the same recursion relation this time with $m \to m+2$, and $m_1 = m+1/2$ (or just substitute $m \to m+1$, in equation above), with rest of the quantum numbers unchanged, and obtain

$$\langle j_1,\frac{1}{2},m+\frac{1}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m+1\rangle = \sqrt{\frac{(j_1+m+\frac{3}{2})}{(j_1+m+\frac{5}{2})}}\langle j_1,\frac{1}{2},m+\frac{3}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m+2\rangle. \tag{7}$$

Using Eq. 7 on the RHS of 6, we obtain

$$\langle j_1,\frac{1}{2},m-\frac{1}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle = \sqrt{\frac{(j_1+m+\frac{1}{2})}{(j_1+m+\frac{3}{2})}}\sqrt{\frac{(j_1+m+\frac{3}{2})}{(j_1+m+\frac{5}{2})}} \times \langle j_1,\frac{1}{2},m+\frac{3}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m+2\rangle. \tag{8}$$

Noting the cancellations above in the numerator and the denominator, after we apply this recursion relation i times, we will have

$$\langle j_1,\frac{1}{2},m-\frac{1}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle = \sqrt{\frac{(j_1+m+\frac{1}{2})}{(j_1+m+\frac{2i+1}{2})}} \times \langle j_1,\frac{1}{2},m+\frac{(2i-1)}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m+i\rangle. \tag{9}$$

This can be continued till the maximum allowed value of i such that $m+i_{max} = j_1+1/2 \implies i_{max} = j_1+\frac{1}{2}-m$ is reached on the RHS, leading to

$$\langle j_1,\frac{1}{2},m-\frac{1}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle = \sqrt{\frac{(j_1+m+\frac{1}{2})}{(2j_1+1)}}\langle j_1,\frac{1}{2},j_1,\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},j_1+\frac{1}{2}\rangle. \tag{10}$$

Because maximum possible values of all the angular momenta are involved, thus, it is obvious

$$|j_1, \frac{1}{2}, j_1, \frac{1}{2}\rangle = |j_1, \frac{1}{2}, j_1 + \frac{1}{2}, j_1 + \frac{1}{2}\rangle$$
$$\implies \langle j_1, \frac{1}{2}, j_1, \frac{1}{2}|j_1, \frac{1}{2}, j_1 + \frac{1}{2}, j_1 + \frac{1}{2}\rangle = 1$$

which, when combined with Eq. 10, gives the desired result

$$\langle j_1, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}|j_1, \frac{1}{2}, j_1 + \frac{1}{2}, m\rangle = \sqrt{\frac{(j_1 + m + \frac{1}{2})}{(2j_1 + 1)}}.$$

We can calculate the magnitude of the CGC $\langle j_1, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}|j_1, \frac{1}{2}, j_1 - \frac{1}{2}, m\rangle$ by using the normalization condition

$$\langle j_1, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}|j_1, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}\rangle = 1$$
$$\implies \sum_{j=j_1-1/2}^{j_1+1/2} \langle j_1, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}|j_1, \frac{1}{2}, j, m\rangle\langle j_1, \frac{1}{2}, j, m|j_1, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}\rangle = 1$$
$$\langle j_1, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}|j_1, \frac{1}{2}, j_1 - \frac{1}{2}, m\rangle^2 + \langle j_1, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}|j_1, \frac{1}{2}, j_1 + \frac{1}{2}, m\rangle^2 = 1$$
$$\implies \langle j_1, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}|j_1, \frac{1}{2}, j_1 - \frac{1}{2}, m\rangle$$
$$= \pm\sqrt{1 - \langle j_1, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}|j_1, \frac{1}{2}, j_1 + \frac{1}{2}, m\rangle^2}$$
$$= \pm\sqrt{\frac{(j_1 - m + \frac{1}{2})}{(2j_1 + 1)}}.$$

We will fix the sign of this CGC in the next part.

(b) $\langle j_1, 1/2, m + 1/2, -1/2|j_1, 1/2, j_1 \pm 1/2, m\rangle = \sqrt{\frac{j_1 \mp m + 1/2}{2j_1+1}}$

Soln: We will first evaluate the CGC $\langle j_1, 1/2, m + 1/2, -1/2|j_1 1/2, j_1 + 1/2, m\rangle$. For the purpose, in the original recursion relation (Eq. 2), we use the upper sign, and set $m \to m - 1$, to obtain the recursion relation

$$\sqrt{(j + m)(j - m + 1)}\langle j_1, j_2, m_1, m_2|j_1, j_2, j, m\rangle =$$
$$\sqrt{(j_1 + m_1)(j_1 - m_1 + 1)}\langle j_1, j_2, m_1 - 1, m_2|j_1, j_2, j, m - 1\rangle$$
$$+\sqrt{(j_2 + m_2)(j_2 - m_2 + 1)}\langle j_1, j_2, m_1, m_2 - 1|j_1, j_2, j, m - 1\rangle$$

In this, if we use $j = j_1 + 1/2$, $j_2 = 1/2$, $m_1 = m + 1/2$, $m_2 = -1/2$, we get

$$\sqrt{(j_1 + m + \frac{1}{2})(j_1 - m + \frac{3}{2})}\langle j_1, \frac{1}{2}, m + \frac{1}{2}, -\frac{1}{2}|j_1, \frac{1}{2}, j_1 + \frac{1}{2}, m\rangle =$$
$$\sqrt{(j_1 + m + \frac{1}{2})(j_1 - m + \frac{1}{2})}\langle j_1, \frac{1}{2}, m - \frac{1}{2}, -\frac{1}{2}|j_1, \frac{1}{2}, j_1 + \frac{1}{2}, m - 1\rangle$$

or

$$\langle j_1,\frac{1}{2},m+\frac{1}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle = \sqrt{\frac{(j_1-m+\frac{1}{2})}{(j_1-m+\frac{3}{2})}}\langle j_1,\frac{1}{2},m-\frac{1}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m-1\rangle$$

which on setting $m \to m-1$ yields

$$\langle j_1,\frac{1}{2},m-\frac{1}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m-1\rangle = \sqrt{\frac{(j_1-m+\frac{3}{2})}{(j_1-m+\frac{5}{2})}}\langle j_1,\frac{1}{2},m-\frac{3}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m-2\rangle$$

so that on substituting this in the previous equation, we obtain (after two iterations)

$$\begin{aligned}\langle j_1,\frac{1}{2},m+\frac{1}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle &= \sqrt{\frac{(j_1-m+\frac{1}{2})}{(j_1-m+\frac{3}{2})}}\sqrt{\frac{(j_1-m+\frac{3}{2})}{(j_1-m+\frac{5}{2})}} \\ &\times \langle j_1,\frac{1}{2},m-\frac{3}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m-2\rangle,\end{aligned}$$

or after i iterations

$$\begin{aligned}\langle j_1,\frac{1}{2},m+\frac{1}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle &= \sqrt{\frac{(j_1-m+\frac{1}{2})}{(j_1-m+\frac{(2i+1)}{2})}} \\ &\times \langle j_1,\frac{1}{2},m-\frac{(2i-1)}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m-i\rangle.\end{aligned}$$

Just, as before, maximum allowed value of i is determined by $m-i_{max} = -j_1-\frac{1}{2} \implies i_{max} = m+j_1+\frac{1}{2}$, leading to

$$\begin{aligned}\langle j_1,\frac{1}{2},m+\frac{1}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle &= \sqrt{\frac{(j_1-m+\frac{1}{2})}{(2j_1+1)}} \\ &\times \langle j_1,\frac{1}{2},-j_1,-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},-j_1-\frac{1}{2}\rangle \\ &= \sqrt{\frac{(j_1-m+\frac{1}{2})}{(2j_1+1)}},\end{aligned}$$

where we used the fact that $\langle j_1,\frac{1}{2},-j_1,-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},-j_1-\frac{1}{2}\rangle = 1$. Using the normalization condition $\langle j_1,\frac{1}{2},m+\frac{1}{2},-\frac{1}{2}|j_1,\frac{1}{2},m+\frac{1}{2},-\frac{1}{2}\rangle = 1$, and inserting the resolution of identity in the coupled representation, as we did in the previous part, we obtain

$$\langle j_1,\frac{1}{2},m+\frac{1}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1-\frac{1}{2},m\rangle = \pm\sqrt{\frac{(j_1+m+\frac{1}{2})}{(2j_1+1)}}$$

To figure out the signs of the two CGCs, we use the orthogonality condition of the coupled states

$$
\begin{aligned}
\langle j_1,\frac{1}{2},j_1+\frac{1}{2},m|j_1,\frac{1}{2},j_1-\frac{1}{2},m\rangle &= 0\\
\implies \sum_{m_1,m_2}\langle j_1,\frac{1}{2},j_1+\frac{1}{2},m|j_1,\frac{1}{2},m_1,m_2\rangle\langle j_1,\frac{1}{2},m_1,m_2|j_1,\frac{1}{2},j_1-\frac{1}{2},m\rangle &= 0\\
\implies \sum_{m_1}\langle j_1,\frac{1}{2},m_1,\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle\langle j_1,\frac{1}{2},m_1,\frac{1}{2}|j_1,\frac{1}{2},j_1-\frac{1}{2},m\rangle&\\
+\sum_{m_1}\langle j_1,\frac{1}{2},m_1,-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle\langle j_1,\frac{1}{2},m_1,-\frac{1}{2}|j_1,\frac{1}{2},j_1-\frac{1}{2},m\rangle &= 0\\
\implies \langle j_1,\frac{1}{2},m-\frac{1}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle\langle j_1,\frac{1}{2},m-\frac{1}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1-\frac{1}{2},m\rangle&\\
+\langle j_1,\frac{1}{2},m+\frac{1}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1+\frac{1}{2},m\rangle\langle j_1,\frac{1}{2},m+\frac{1}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1-\frac{1}{2},m\rangle &= 0.
\end{aligned}
\tag{11}
$$

Let us use the notations $\langle j_1,\frac{1}{2},m-\frac{1}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1\pm\frac{1}{2},m\rangle = x_1/y_1$ and $\langle j_1,\frac{1}{2},m+\frac{1}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1\pm\frac{1}{2},m\rangle = x_2/y_2$. There is no ambiguity about the signs of x_1/x_2, both of which have been calculated to be positive. Eq. 11 is

$$
\begin{aligned}
x_1y_1+x_2y_2 &= 0\\
y_1 &= -\frac{x_2}{x_1}y_2
\end{aligned}
$$

This implies that sign of y_1 will be opposite to that of y_2. So if we take y_2 to be positive

$$
y_2 = \langle j_1,\frac{1}{2},m+\frac{1}{2},-\frac{1}{2}|j_1,\frac{1}{2},j_1-\frac{1}{2},m\rangle = \sqrt{\frac{(j_1+m+\frac{1}{2})}{(2j_1+1)}},
$$

y_1 will be negative

$$
y_1 = \langle j_1,\frac{1}{2},m-\frac{1}{2},\frac{1}{2}|j_1,\frac{1}{2},j_1-\frac{1}{2},m\rangle = -\sqrt{\frac{(j_1-m+\frac{1}{2})}{(2j_1+1)}}
$$

3. Show that the eigenvectors of total angular momentum $\mathbf{J}$, obtained by coupling the orbital angular momentum (l) and the spin angular momentum ($s=1/2$) of an electron can be written as

$$
\mathcal{Y}_l^{jm}(\theta,\phi) = \mathcal{Y}_l^{l\pm1/2,m}(\theta,\phi) = \frac{1}{\sqrt{2l+1}}\begin{pmatrix} \pm\sqrt{l\pm m+\frac{1}{2}}Y_l^{m-1/2}(\theta,\phi)\\ \sqrt{l\mp m+\frac{1}{2}}Y_l^{m+1/2}(\theta,\phi)\end{pmatrix}.
$$

Also verify that $\mathcal{Y}_l^{jm}$ is an eigenfunction of $\mathbf{J}^2$ operator, where $\mathbf{J}=\mathbf{L}+\mathbf{S}$.
Soln: Here we have to couple the eigenfunctions of $\mathbf{L}^2$ operator, $|lm_l\rangle$, and those of

spin operator $\mathbf{S}^2$, $|\frac{1}{2}m_s\rangle$ (with $m_s = \pm\frac{1}{2}$) to obtain the coupled states $|l\frac{1}{2}jm\rangle$, with $j = l \pm \frac{1}{2}$, which are eigenfunctions of $\mathbf{J}^2$ operator. Obviously

$$
\begin{aligned}
|l, \frac{1}{2}, j, m\rangle = & \sum_{m_l, m_s} \langle l, \frac{1}{2}, m_l, m_s | l, \frac{1}{2}, j, m\rangle | l, \frac{1}{2}, m_l, m_s\rangle \\
= & \sum_{m_l} \langle l, \frac{1}{2}, m_l, \frac{1}{2} | l, \frac{1}{2}, j, m\rangle | l, \frac{1}{2}, m_l, \frac{1}{2}\rangle + \sum_{m_l} \langle l, \frac{1}{2}, m_l, -\frac{1}{2} | l, \frac{1}{2}, j, m\rangle | l, \frac{1}{2}, m_l, -\frac{1}{2}\rangle \\
= & \langle l, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2} | l, \frac{1}{2}, j, m\rangle | l, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}\rangle + \langle l, \frac{1}{2}, m + \frac{1}{2}, -\frac{1}{2} | l, \frac{1}{2}, j, m\rangle | l, \frac{1}{2}, m + \frac{1}{2}, -\frac{1}{2}\rangle
\end{aligned}
$$

CGCs required in this equation were computed in the previous problem, and can be obtained by setting $j_1 = l$, and $j = l \pm \frac{1}{2}$. Thus

$$
\begin{aligned}
|l, \frac{1}{2}, j, m\rangle \equiv & |l, \frac{1}{2}, l \pm \frac{1}{2}, m\rangle \\
= & \pm\sqrt{\frac{l \pm m + 1/2}{2l + 1}} |l, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}\rangle + \sqrt{\frac{l \mp m + 1/2}{2l + 1}} |l, \frac{1}{2}, m + \frac{1}{2}, -\frac{1}{2}\rangle \quad (12)
\end{aligned}
$$

This is the desired result in the Dirac notation. To express this result in the $|\mathbf{x}\rangle = |\mathbf{r}\rangle \otimes |\sigma\rangle = |\mathbf{r}\sigma\rangle$, we take the inner product of the equation above with the understanding

$$
\langle \mathbf{x} | l, \frac{1}{2}, j, m\rangle \equiv \mathcal{Y}_l^{jm}(\theta, \phi) \equiv \mathcal{Y}_l^{l\pm 1/2, m}(\theta, \phi),
$$

and

$$
\langle \mathbf{x} | l, \frac{1}{2}, m', \pm\frac{1}{2}\rangle = \langle \mathbf{r} | l m'\rangle \langle \sigma | \frac{1}{2}, \pm\frac{1}{2}\rangle = Y_l^{m'}(\theta, \phi)\alpha/\beta,
$$

$$
\text{where } \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

With this in Eq. 12, the LHS is

$$
\langle \mathbf{x} | l, \frac{1}{2}, j, m\rangle = \langle \mathbf{x} | l, \frac{1}{2}, l \pm \frac{1}{2}, m\rangle = \mathcal{Y}_l^{jm}(\theta, \phi) = \mathcal{Y}_l^{l\pm 1/2, m}(\theta, \phi)
$$

while the RHS becomes

$$
\begin{aligned}
& = \pm\sqrt{\frac{l \pm m + \frac{1}{2}}{2l + 1}} \langle \mathbf{x} | l, \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2}\rangle + \sqrt{\frac{l \mp m + \frac{1}{2}}{2l + 1}} \langle \mathbf{x} | l, \frac{1}{2}, m + \frac{1}{2}, -\frac{1}{2}\rangle \\
& = \pm\sqrt{\frac{l \pm m + \frac{1}{2}}{2l + 1}} Y_l^{m-1/2}(\theta, \phi)\alpha + \sqrt{\frac{l \mp m + \frac{1}{2}}{2l + 1}} Y_l^{m+1/2}(\theta, \phi)\beta \\
& = \frac{1}{\sqrt{2l + 1}} \begin{pmatrix} \pm\sqrt{l \pm m + \frac{1}{2}} Y_l^{m-1/2}(\theta, \phi) \\ \sqrt{l \mp m + \frac{1}{2}} Y_l^{m+1/2}(\theta, \phi) \end{pmatrix}
\end{aligned}
$$

leading to the final result

$$
\mathcal{Y}_l^{jm}(\theta, \phi) \equiv \mathcal{Y}_l^{l\pm 1/2, m}(\theta, \phi) = \frac{1}{\sqrt{2l + 1}} \begin{pmatrix} \pm\sqrt{l \pm m + \frac{1}{2}} Y_l^{m-1/2}(\theta, \phi) \\ \sqrt{l \mp m + \frac{1}{2}} Y_l^{m+1/2}(\theta, \phi) \end{pmatrix}.
$$

To show that $\mathcal{Y}_l^{jm}$ are eigenfunctions of $\mathbf{J}^2$ operator, we note

$$\mathbf{J}^2 = (\mathbf{L}+\mathbf{S})^2 = \mathbf{L}^2 + \mathbf{S}^2 + 2\mathbf{L}\cdot\mathbf{S}.$$

It is easy to evaluate the action of $\mathbf{L}^2$ and $\mathbf{S}^2$ on $\mathcal{Y}_l^{jm}(\theta,\phi)$,because

$$\begin{aligned}\mathbf{L}^2 Y_l^m(\theta,\phi) &= l(l+1)\hbar^2 Y_l^m(\theta,\phi)\\ \mathbf{S}^2(\alpha/\beta) &= \frac{3}{4}\hbar^2(\alpha/\beta).\end{aligned}$$

With this

$$\begin{aligned}\mathbf{L}^2\mathcal{Y}_l^{jm}(\theta,\phi) &= l(l+1)\hbar^2\mathcal{Y}_l^{jm}(\theta,\phi)\\ \mathbf{S}^2\mathcal{Y}_l^{jm}(\theta,\phi) &= \frac{3}{4}\hbar^2\mathcal{Y}_l^{jm}(\theta,\phi)\end{aligned}$$

To compute the action of $\mathbf{L}\cdot\mathbf{S}$ operator, we write it as

$$\begin{aligned}\mathbf{L}\cdot\mathbf{S} &= L_xS_x + L_yS_y + L_zS_z\\ &= \frac{\hbar}{2}\begin{pmatrix} L_z & L_- \\ L_+ & -L_z\end{pmatrix},\end{aligned}$$

so that

$$\begin{aligned}2\mathbf{L}\cdot\mathbf{S}\mathcal{Y}_l^{jm}(\theta,\phi) &= 2\mathbf{L}\cdot\mathbf{S}\mathcal{Y}_l^{l\pm1/2,m}(\theta,\phi)\frac{\hbar}{\sqrt{2l+1}}\begin{pmatrix} L_z & L_- \\ L_+ & -L_z\end{pmatrix}\begin{pmatrix} \pm\sqrt{l\pm m+\frac{1}{2}}Y_l^{m-1/2}(\theta,\phi) \\ \sqrt{l\mp m+\frac{1}{2}}Y_l^{m+1/2}(\theta,\phi)\end{pmatrix}\\ &= \frac{\hbar}{\sqrt{2l+1}}\begin{pmatrix} \pm\sqrt{l\pm m+\frac{1}{2}}L_zY_l^{m-1/2}(\theta,\phi) + \sqrt{l\mp m+\frac{1}{2}}L_-Y_l^{m+1/2}(\theta,\phi) \\ \pm\sqrt{l\pm m+\frac{1}{2}}L_+Y_l^{m-1/2}(\theta,\phi) - \sqrt{l\mp m+\frac{1}{2}}L_zY_l^{m+1/2}(\theta,\phi)\end{pmatrix}.\end{aligned} \tag{13}$$

Using the relations

$$\begin{aligned}L_zY_l^m(\theta,\phi) &= m\hbar Y_l^m(\theta,\phi)\\ L_\pm Y_l^m(\theta,\phi) &= \hbar\sqrt{l(l+1)-m(m\pm1)}Y_l^{m\pm1}(\theta,\phi),\end{aligned}$$

we obtain for the first (upper) component of the spinor on the RHS of Eq. 13

$$\begin{aligned}&\pm\sqrt{l\pm m+\frac{1}{2}}L_zY_l^{m-1/2}(\theta,\phi) + \sqrt{l\mp m+\frac{1}{2}}L_-Y_l^{m+1/2}(\theta,\phi)\\ &= \pm\hbar\sqrt{l\pm m+\frac{1}{2}}(m-\frac{1}{2})Y_l^{m-1/2} + \hbar\sqrt{l\mp m+\frac{1}{2}}\sqrt{l(l+1)-(m+\frac{1}{2})(m-\frac{1}{2})}Y_l^{m-1/2}\end{aligned}$$

We use the fact that $\sqrt{l(l+1)-(m+\frac{1}{2})(m-\frac{1}{2})} = \sqrt{(l+m+\frac{1}{2})(l-m+\frac{1}{2})}$ to simplify

$$\pm\sqrt{l\pm m+\frac{1}{2}}L_zY_l^{m-1/2}(\theta,\phi)+\sqrt{l\mp m+\frac{1}{2}}L_-Y_l^{m+1/2}(\theta,\phi)$$
$$=\pm\hbar\sqrt{l\pm m+\frac{1}{2}}Y_l^{m-1/2}\left(m-\frac{1}{2}\pm\frac{\sqrt{l\mp m+\frac{1}{2}}}{\sqrt{l\pm m+\frac{1}{2}}}\sqrt{(l+m+\frac{1}{2})(l-m+\frac{1}{2})}\right)$$
$$=\pm\hbar\sqrt{l\pm m+\frac{1}{2}}Y_l^{m-1/2}\,\{l\text{ (for upper sign) and }-(l+1)\text{ (for lower sign)}\}$$

Similarly, we can show for the second (lower) component of the spinor on the RHS of Eq. 13

$$\pm\sqrt{l\pm m+\frac{1}{2}}L_+Y_l^{m-1/2}(\theta,\phi)-\sqrt{l\mp m+\frac{1}{2}}L_zY_l^{m+1/2}(\theta,\phi)$$
$$=\hbar\sqrt{l\mp m+\frac{1}{2}}Y_l^{m+1/2}(\theta,\phi)\,\{l\text{ (for upper sign) and }-(l+1)\text{ (for lower sign)}\}.$$

But, it is easy to see that the factor common to both the upper and the lower components can be simplified as

$$\{l\text{ (for upper sign) and }-(l+1)\text{ (for lower sign)}\}=\pm(l+\frac{1}{2})-\frac{1}{2}$$

As a result we have

$$2\mathbf{L}\cdot\mathbf{S}\mathcal{Y}_l^{l\pm\frac{1}{2},m}(\theta,\phi)=(\pm(l+\frac{1}{2})-\frac{1}{2})\mathcal{Y}_l^{l\pm\frac{1}{2},m}(\theta,\phi).$$

Combining all the terms, we obtain

$$\begin{aligned}\mathbf{J}^2\mathcal{Y}_l^{l\pm\frac{1}{2},m}(\theta,\phi)&=(\mathbf{L}^2+\mathbf{S}^2+2\mathbf{L}\cdot\mathbf{S})\mathcal{Y}_l^{l\pm\frac{1}{2},m}(\theta,\phi)\\&=\hbar^2\{l(l+1)+\frac{3}{4}+(\pm(l+\frac{1}{2})-\frac{1}{2})\}\mathcal{Y}_l^{l\pm\frac{1}{2},m}(\theta,\phi)\\&=\hbar^2(l\pm\frac{1}{2})(l\pm\frac{1}{2}+1)\mathcal{Y}_l^{l\pm\frac{1}{2},m}(\theta,\phi).\end{aligned}$$

On substituting $j=l\pm\frac{1}{2}$ above, we obtain the desired relation

$$\mathbf{J}^2\mathcal{Y}_l^{jm}(\theta,\phi)=\hbar^2j(j+1)\mathcal{Y}_l^{jm}(\theta,\phi).$$

Note: Try to verify by explicit calculations that $J_z\mathcal{Y}_l^{jm}(\theta,\phi)=m\hbar\mathcal{Y}_l^{jm}(\theta,\phi)$. For the purpose use $J_z=L_z+S_z\equiv L_z\otimes I_s+I_l\times S_z$, and note that $I_l$is just number 1, while I_s is a 2×2 identity matrix.

4. Suppose you have two spin $\frac{1}{2}$ particles, with their individual spin operators $\mathbf{S_1}$ and $\mathbf{S}_2$. Obtain the eigenstates of $\mathbf{S^2}$ and $\mathbf{S}_z$ operators, where $\mathbf{S}=\mathbf{S}_1+\mathbf{S}_2$, by the following two approaches:

(a) Using the C-G coefficients

Soln: When we couple two spin 1/2 particles, the possible values of coupled angular momenta J as per triangular inequality are $0 \leq J \leq 1$, i.e., $J = 0, 1$, for the coupled states. Because in this problem both $j_1 = j_2 = 1/2$, we define a shorter notation in which values of j_1 and j_2 are suppressed, as follows

$$|j_1, j_2, j, m\rangle \equiv |1/2, 1/2, j, m\rangle \equiv |jm\rangle$$
$$|j_1, j_2, m_1, m_2\rangle \equiv |1/2, 1/2, m_1, m_2\rangle \equiv |m_1, m_2\rangle$$
$$\langle j_1, j_2, m_1, m_2|j_1, j_2, j, m\rangle = \langle 1/2, 1/2, m_1, m_2|1/2, 1/2, j, m\rangle \equiv \langle m_1, m_2|jm\rangle$$

Here $|jm\rangle = |0,0\rangle$, $|1,-1\rangle$, $|1,0\rangle$ and, $|1,1\rangle$. The uncoupled states in $|m_1, m_2\rangle$ format are $|\frac{1}{2}, \frac{1}{2}\rangle$, $|\frac{1}{2}, -\frac{1}{2}\rangle$, $|-\frac{1}{2}, \frac{1}{2}\rangle$, $|-\frac{1}{2}, -\frac{1}{2}\rangle$. Using the compact notation for CGCs $\langle m_1, m_2|jm\rangle$ defined above, we have

$$|0,0\rangle = \langle \frac{1}{2}, -\frac{1}{2}|00\rangle|\frac{1}{2}, -\frac{1}{2}\rangle + \langle -\frac{1}{2}, \frac{1}{2}|00\rangle| - \frac{1}{2}, \frac{1}{2}\rangle.$$

Taking $j_1 = \frac{1}{2}$, we can compute the required CGCs from the results of problem 2

$$\langle \frac{1}{2}, -\frac{1}{2}|00\rangle = \frac{1}{\sqrt{2}}$$
$$\langle -\frac{1}{2}, \frac{1}{2}|00\rangle = -\frac{1}{\sqrt{2}},$$

so that

$$|0,0\rangle = \frac{1}{\sqrt{2}}|\frac{1}{2}, -\frac{1}{2}\rangle - \frac{1}{\sqrt{2}}| - \frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{2}}\{\alpha(1)\beta(2) - \beta(1)\alpha(2)\}.$$

Similarly

$$|1,-1\rangle = \langle -\frac{1}{2}, -\frac{1}{2}|1-1\rangle| - \frac{1}{2}, -\frac{1}{2}\rangle$$
$$|1,0\rangle = \langle \frac{1}{2}, -\frac{1}{2}|10\rangle|\frac{1}{2}, -\frac{1}{2}\rangle + \langle -\frac{1}{2}, \frac{1}{2}|10\rangle| - \frac{1}{2}, \frac{1}{2}\rangle$$
$$|1,1\rangle = \langle \frac{1}{2}, \frac{1}{2}|11\rangle|\frac{1}{2}, \frac{1}{2}\rangle.$$

Using the formulas of Prob 2 above

$$\langle -\frac{1}{2}, -\frac{1}{2}|1-1\rangle = \langle \frac{1}{2}, \frac{1}{2}|11\rangle = 1$$
$$\langle \frac{1}{2}, -\frac{1}{2}|10\rangle = \langle -\frac{1}{2}, \frac{1}{2}|10\rangle = \frac{1}{\sqrt{2}}$$

we have

$$|1,-1\rangle = | - \frac{1}{2}, -\frac{1}{2}\rangle = \beta(1)\beta(2)$$
$$|1,0\rangle = \frac{1}{\sqrt{2}}|\frac{1}{2}, -\frac{1}{2}\rangle + \frac{1}{\sqrt{2}}| - \frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{2}}\{\alpha(1)\beta(2) + \beta(1)\alpha(2)\}$$
$$|1,1\rangle = |\frac{1}{2}, \frac{1}{2}\rangle = \alpha(1)\alpha(2).$$

(b) By constructing the $\mathbf{S}^2$ operator in the uncoupled basis, and diagonalizing it.
Soln: We will construct the $\mathbf{S}^2$ operator in the uncoupled basis as follows

$$\begin{aligned}\mathbf{S}^2 &= (\mathbf{S}_1+\mathbf{S}_2)^2 = \mathbf{S}_1^2+\mathbf{S}_2^2+2\mathbf{S}_1\cdot\mathbf{S}_2\\ &= \mathbf{S}_1^2\otimes I_2 + I_1\otimes\mathbf{S}_2^2 + 2S_{x1}\otimes S_{x2} + 2S_{y1}\otimes S_{y2} + 2S_{z1}\otimes S_{z2}.\end{aligned}$$

Using the fact that $\mathbf{S}_1^2 = \mathbf{S}_2^2 = \frac{3\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix} = \frac{3\hbar^2}{4}I_{1/2}$, and the standard representations of S_x, S_y, and S_z, we obtain

$$\begin{aligned}\mathbf{S}^2 &= \frac{3\hbar^2}{4}I_1\otimes I_2 + \frac{3\hbar^2}{4}I_2\otimes I_2 + \frac{\hbar^2}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix} + \frac{\hbar^2}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\otimes\begin{pmatrix}0&-i\\i&0\end{pmatrix}\\
&+\frac{\hbar^2}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix}\\
&= \frac{3\hbar^2}{2}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} + \frac{\hbar^2}{2}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\\
&+\frac{\hbar^2}{2}\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix} + \frac{\hbar^2}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\\
&= \hbar^2\begin{pmatrix}2&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&2\end{pmatrix}.\end{aligned}$$

Above, the ordered basis is $\left\{|\frac{1}{2},\frac{1}{2}\rangle,\ |\frac{1}{2},-\frac{1}{2}\rangle,\ |-\frac{1}{2},\frac{1}{2}\rangle,\ |-\frac{1}{2},-\frac{1}{2}\rangle\right\}$. To diagonalize $\mathbf{S}^2$, we obtain its eigenvalues and eigenvectors. The characteristic polynomial of $\mathbf{S}^2$ is

$$\begin{aligned}|\mathbf{S}^2-\lambda I| = \hbar^2\begin{vmatrix}2-\lambda&0&0&0\\0&1-\lambda&1&0\\0&1&1-\lambda&0\\0&0&0&2-\lambda\end{vmatrix} &= 0\\
= (2-\lambda)^2\{(1-\lambda)^2-1\} = \lambda(\lambda-2)^3 &= 0\\
\implies \lambda = 0, 2, 2, 2.&\end{aligned}$$

Because the eigenvalue of $\mathbf{S}^2$ will be of the form $s(s+1)$, so $\lambda = 0 \implies s = 0$ (singlet) and $\lambda = 2 \implies s = 1$ (triplet). Let us find the eigenvectors from the homogeneous equation $(\mathbf{S}^2-\lambda I)C = 0$, where $C = \begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix}$ denotes the eigenvector. For $\lambda = 0$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0.$$

This leads to

$$c_1 = c_4 = 0$$
$$c_2 + c_3 = 0$$

which leads to the normalized solution

$$|\lambda = 0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}|\frac{1}{2}, -\frac{1}{2}\rangle - \frac{1}{\sqrt{2}}|-\frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{2}} \left\{\alpha(1)\beta(2) - \beta(1)\alpha(2)\right\}.$$

For $\lambda = 2$, we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0.$$

This leaves c_1 and c_4 uncertain, while $c_2 - c_3 = 0$. This allows three linearly independent solutions

$$|\lambda = 2\rangle_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ |\lambda = 2\rangle_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ |\lambda = 2\rangle_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

which can, respectively be written in the uncoupled basis as

$$|\lambda = 2\rangle_1 = |\frac{1}{2}, \frac{1}{2}\rangle = |11\rangle$$
$$|\lambda = 2\rangle_2 = \frac{1}{\sqrt{2}}|\frac{1}{2}, -\frac{1}{2}\rangle + \frac{1}{\sqrt{2}}| - \frac{1}{2}, \frac{1}{2}\rangle = |10\rangle$$
$$|\lambda = 2\rangle_3 = | - \frac{1}{2}, -\frac{1}{2}\rangle = |1 - 1\rangle$$

5. Calculate the C-G coefficients needed to couple the two angular momenta $j_1 = 3/2$ and $j_2 = 1$ to the possible j values, and express the coupled states $|j_1 j_2 j m\rangle$ in terms of the uncoupled state $|j_1 j_2 m_1 m_2\rangle$.
Soln: Allowed values of coupled angular momenta according to the triangular inequality are $j = 1/2,\ 3/2,\ 5/2$. Let us construct the coupled states for each of those values.

(i)$j = 1/2$
We can write $|\frac{1}{2}m\rangle$,$m = \pm\frac{1}{2}$ as

$$\begin{aligned}
|\frac{1}{2},\frac{1}{2}\rangle &= \sum_{m_1,m_2} \langle m_1, m_2|\frac{1}{2},\frac{1}{2}\rangle|m_1, m_2\rangle \\
&= \sum_{m_1}\langle m_1, -1|\frac{1}{2},\frac{1}{2}\rangle|m_1, -1\rangle + \sum_{m_1}\langle m_1, 0|\frac{1}{2},\frac{1}{2}\rangle|m_1, 0\rangle + \sum_{m_1}\langle m_1 1|\frac{1}{2},\frac{1}{2}\rangle|m_1, 1\rangle \\
&= \langle\frac{3}{2}, -1|\frac{1}{2},\frac{1}{2}\rangle|\frac{3}{2}, -1\rangle + \langle\frac{1}{2}, 0|\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2}, 0\rangle + \langle -\frac{1}{2}, 1|\frac{1}{2},\frac{1}{2}\rangle| -\frac{1}{2}, 1\rangle.
\end{aligned}$$

One can compute

$$\begin{aligned}
\langle\frac{3}{2}, -1|\frac{1}{2},\frac{1}{2}\rangle &= \frac{1}{\sqrt{2}} \\
\langle\frac{1}{2}, 0|\frac{1}{2},\frac{1}{2}\rangle &= -\frac{1}{\sqrt{3}} \\
\langle -\frac{1}{2}, 1|\frac{1}{2},\frac{1}{2}\rangle &= \frac{1}{\sqrt{6}},
\end{aligned}$$

so that

$$|\frac{1}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}|\frac{3}{2}, -1\rangle - \frac{1}{\sqrt{3}}|\frac{1}{2}, 0\rangle + \frac{1}{\sqrt{6}}| -\frac{1}{2}, 1\rangle.$$

Similarly

$$|\frac{1}{2}, -\frac{1}{2}\rangle = \langle -\frac{3}{2}, 1|\frac{1}{2}, -\frac{1}{2}\rangle| -\frac{3}{2}, 1\rangle + \langle -\frac{1}{2}, 0|\frac{1}{2}, -\frac{1}{2}\rangle| -\frac{1}{2}, 0\rangle + \langle\frac{1}{2}, -1|\frac{1}{2}, -\frac{1}{2}\rangle|\frac{1}{2}, -1\rangle$$

Using the symmetry property of CGCs

$$\langle m_1, m_2|j, m\rangle = (-1)^{j-j_1-j_2}\langle -m_1, -m_2|j, -m\rangle,$$

we obtain

$$|\frac{1}{2}, -\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}| -\frac{3}{2}, 1\rangle - \frac{1}{\sqrt{3}}| -\frac{1}{2}, 0\rangle + \frac{1}{\sqrt{6}}|\frac{1}{2}, -1\rangle.$$

(ii) $j = 3/2$
We can write $|\frac{3}{2}m\rangle$,$m = \pm\frac{1}{2}, \pm\frac{3}{2}$ as

$$\begin{aligned}
|\frac{3}{2},\frac{3}{2}\rangle &= \sum_{m_1,m_2} \langle m_1, m_2|\frac{3}{2},\frac{3}{2}\rangle|m_1, m_2\rangle \\
&= \sum_{m_1}\langle m_1, -1|\frac{3}{2},\frac{3}{2}\rangle|m_1, -1\rangle + \sum_{m_1}\langle m_1, 0|\frac{3}{2},\frac{3}{2}\rangle|m_1, 0\rangle + \sum_{m_1}\langle m_1 1|\frac{3}{2},\frac{3}{2}\rangle|m_1, 1\rangle \\
&= \langle\frac{3}{2}, 0|\frac{3}{2},\frac{3}{2}\rangle|\frac{3}{2}, 0\rangle + \langle\frac{1}{2}, 1|\frac{3}{2},\frac{3}{2}\rangle|\frac{1}{2}, 1\rangle \\
&= \sqrt{\frac{3}{5}}|\frac{3}{2}, 0\rangle - \sqrt{\frac{2}{5}}|\frac{1}{2}, 1\rangle.
\end{aligned}$$

Using the symmetry of CGCs, we obtain

$$|\frac{3}{2},-\frac{3}{2}\rangle = -\sqrt{\frac{3}{5}}|-\frac{3}{2},0\rangle + \sqrt{\frac{2}{5}}|-\frac{1}{2},-1\rangle.$$

The two results can be combined as

$$|\frac{3}{2},\pm\frac{3}{2}\rangle = \pm\sqrt{\frac{3}{5}}|\pm\frac{3}{2},0\rangle \mp \sqrt{\frac{2}{5}}|\pm\frac{1}{2},\pm1\rangle.$$

Similarly

$$\begin{aligned}|\frac{3}{2},\pm\frac{1}{2}\rangle &= \langle\pm\frac{3}{2},\mp1|\frac{3}{2},\pm\frac{1}{2}\rangle|\pm\frac{3}{2},\mp1\rangle + \langle\pm\frac{1}{2},0|\frac{3}{2},\pm\frac{1}{2}\rangle|\pm\frac{1}{2},0\rangle \\ &+ \langle\mp\frac{1}{2},\pm1|\frac{3}{2},\pm\frac{1}{2}\rangle|\mp\frac{1}{2},\pm1\rangle \\ &= \pm\sqrt{\frac{2}{5}}|\pm\frac{3}{2},\mp1\rangle \pm \sqrt{\frac{1}{15}}|\pm\frac{1}{2},0\rangle \mp \sqrt{\frac{8}{15}}|\mp\frac{1}{2},\pm1\rangle\end{aligned}$$

(iii)$j = 5/2$
For $m = \pm\frac{5}{2}$

$$|\frac{5}{2},\pm\frac{5}{2}\rangle = \langle\pm\frac{3}{2},\pm1|\frac{5}{2},\pm\frac{5}{2}\rangle|\pm\frac{3}{2},\pm1\rangle = |\pm\frac{3}{2},\pm1\rangle$$

For $m = \pm\frac{3}{2}$

$$\begin{aligned}|\frac{5}{2},\pm\frac{3}{2}\rangle &= \langle\pm\frac{3}{2},0|\frac{5}{2},\pm\frac{3}{2}\rangle|\pm\frac{3}{2},0\rangle + \langle\pm\frac{1}{2},\pm1|\frac{5}{2},\pm\frac{3}{2}\rangle|\pm\frac{1}{2},\pm1\rangle \\ &= \sqrt{\frac{3}{5}}|\pm\frac{3}{2},0\rangle - \sqrt{\frac{2}{5}}|\pm\frac{1}{2},\pm1\rangle\end{aligned}$$

For $m = \pm\frac{1}{2}$

$$\begin{aligned}|\frac{5}{2},\pm\frac{1}{2}\rangle &= \langle\pm\frac{3}{2},\mp1|\frac{5}{2},\pm\frac{1}{2}\rangle|\pm\frac{3}{2},\mp1\rangle + \langle\pm\frac{1}{2},0|\frac{5}{2},\pm\frac{1}{2}\rangle|\pm\frac{1}{2},0\rangle + \langle\mp\frac{1}{2},\pm1|\frac{5}{2},\pm\frac{1}{2}\rangle|\mp\frac{1}{2},\pm1\rangle \\ &= \sqrt{\frac{1}{10}}|\pm\frac{3}{2},\mp1\rangle + \sqrt{\frac{3}{5}}|\pm\frac{1}{2},0\rangle + \sqrt{\frac{3}{10}}|\mp\frac{1}{2},\pm1\rangle\end{aligned}$$