# PH 422: Quantum Mechanics II
# Tutorial Sheet 2

This tutorial sheet contains problems related to the Clebsch-Gordon series, tensor operators, and the Wigner-Eckart theorem.

1. By use of the unitary condition for the $D$ matrices and the orthogonality condition of the CGCs, derive from the Clebsch-Gordon series the result

$$\sum_{m_2'} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j_3 m_3 \rangle D^{(j_2)}_{m_2' m_2}(R) = \sum_{m_1 m} D^{(j_3)}_{m_3 m}(R) \langle j_1 j_2 m_1 m_2 | j_1 j_2 j_3 m \rangle D^{(j_1)*}_{m_1' m_1}(R).$$

2. Prove

$$D^{(j)}_{mm'}(R) = \sum_{m_1, m_1', m_2, m_2'} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle D^{(j_1)}_{m_1 m_1'}(R) D^{(j_2)}_{m_2, m_2'}(R) \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m' \rangle.$$

   Verify that this result holds for $j_1 = 1/2$, $j_2 = 1$, $j = 3/2$, when $R$ denotes a rotation by an angle $\theta$ about the $z$ axis.

3. Using the spherical harmonics $Y_l^m(\theta, \phi)$, establish the connection between components of a Cartesian tensor of rank 2 defined as $T_{ij} = x_i x_j$, and the corresponding spherical tensor $T^q_{k=2}$. Here $x_i$ denotes the $i$-th Cartesian component of the position vector.

4. If $|nlm\rangle$ denotes an eigenfunction of the hydrogen atom (without considering its spin), and we define

$$\chi = \langle n' = 3, l' = 2, m' = 2 | xy | n = 3, l = 0, m = 0 \rangle.$$

   Compute, as a function of $\chi$, the matrix elements

$$\langle n' = 3, l' = 2, m' | T_{ij} | n = 3, l = 0, m = 0 \rangle,$$

   where $T_{ij}$ is defined in the previous problem.

5. Directly compute the matrix elements

$$\langle j = 1, m' | J^q | j = 1, m \rangle,$$

   where $J^q$ denotes the $q$-th spherical component of the angular momentum operator. Verify that these matrix elements satisfy Wigner-Eckart theorem, and deduce the corresponding reduced matrix elements from them.

6. Evaluate

$$\sum_{m=-j}^{j} |D^{(j)}_{mm'}(\beta)|^2 m$$

   for any $j$ (integer or half-integer), then check your answer for $j = \frac{1}{2}$.

7. Prove the following results for $j = 1$, using the corresponding representation of $J_y$

   (a)

$$e^{-\frac{iJ_y\beta}{\hbar}} = 1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta)$$

   (b)

$$D^{(j=1)}(\beta) = \begin{pmatrix} \left(\frac{1}{2}\right)(1+\cos\beta) & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1-\cos\beta) \\ \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \cos\beta & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta \\ \left(\frac{1}{2}\right)(1-\cos\beta) & \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1+\cos\beta) \end{pmatrix}$$

8. Consider a spherical tensor of rank 1 (that is, a vector)

$$V_1^{\pm 1} = \mp\frac{V_x \pm iV_y}{\sqrt{2}}, \qquad V_1^0 = V_z.$$

   Using the expression for $D^{(j=1)}(\beta)$ given in the previous problem, evaluate

$$\sum_{q'} D^{(1)}_{qq'}(\beta)V_1^{q'},$$

   and show that your results are just what you expect from the transformation properties of $V_{x,y,z}$, under rotation about the $y$-axis.

9. (a) Construct a spherical tensor of rank 1 out of two different vectors $\mathbf{U} = (U_x, U_y, U_z)$ and $\mathbf{V} = (V_x, V_y, V_z)$. Explicitly write $T_1^{\pm 1,0}$, in terms of $U_{x,y,z}$ and $V_{x,y,z}$.

   (b) Construct a spherical tensor of rank 2 out of two different vectors $\mathbf{U}$ and $\mathbf{V}$. Write down explicitly $T_2^{\pm 2,\pm 1,0}$, in terms of $U_{x,y,z}$ and $V_{x,y,z}$.

10. Consider a spinless particle bound to a fixed center by a central force potential.

    (a) Relate, as much as possible, the matrix elements

$$\langle n', l', m'| \mp \frac{1}{\sqrt{2}}(x \pm iy)|n, l, m\rangle \qquad \text{and } \langle n', l', m'|z|n, l, m\rangle$$

    using only the Wigner-Eckart theorem. Make sure to state under what conditions the matrix elements are nonvanishing.

    (b) Do the same problem using the wave function $\psi_{nlm}(\mathbf{r}) = R_{nl}(r)Y_l^m(\theta, \phi)$.

11. (a) Write $xy$, $xz$, and $(x^2 - y^2)$ as components of a spherical (irreducible) tensor of rank 2.

(b) The expectation value

$$Q \equiv e\langle \alpha, j, m = j | (3z^2 - r^2) | \alpha, j, m = j \rangle$$

is known as *quadrupole moment.* Evaluate

$$e\langle \alpha, j, m^{'} | (x^2 - y^2) | \alpha, j, m = j \rangle,$$

(where $m^{'} = j, j-1, j-2, \ldots$) in terms of $Q$ and appropriate C-G coefficients.

12. The magnetic moment of an atom is defined as

$$\boldsymbol{\mu} = -\frac{e}{2mc}\left(g_L \mathbf{L} + g_S \mathbf{S}\right),$$

where $e$ is the electronic charge, $m$ is the electronic mass, $c$ is the speed of light, $\mathbf{L}$ is the total orbital angular momentum operator for the atom, $\mathbf{S}$ is total spin angular momentum operator of the atom, and $g_L$ and $g_S$ are, respectively, orbital and spin Lande g factors.

(a) Argue that the expectation value components $\langle \mu_i \rangle = \langle \alpha j j | \mu_i | \alpha j j \rangle$, are proportional to each other

(b) Using the projection theorem, prove that if $g_L = 1$ and $g_S = 2$, $\mu = \langle \mu_z \rangle$ is given by

$$\mu = -\frac{e\hbar}{2mc} g_J J,$$

where

$$g_J = \left\{1 + \frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)}\right\},$$

and $S$, $L$, and $J$, respectively denote the total spin, orbital angular momentum, and total angular momentum of the atom.