Chapter 1: Angular Momentum Algebra II

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Summary of the Chapter

In this chapter, first we will briefly review what you have learnt in angular momentum algebra in the first part of this course. After that, we will discuss rotation operators and their representations. The theory will be developed for rotations about general axes, and will make use of the Euler angles. Next important topic will be the addition of angular momenta using the Clebsch-Gordon methodology. For the purpose, the concept of tensor product spaces will be introduced, and the operations of direct sum and direct products will be defined. Using the theory developed, Wigner-Eckart theorem, and its corollary, projection theorem will be proved. Finally, the applications of these concepts will be discussed in various problems in the tutorial sheets.

Introduction

- You studied the basics of angular momentum algebra in the previous course (Q. Mech. I), last semester
- We will first briefly review that
- This will be followed by a discussion of rotation operators and their representations
- Next, we will introduce the concept of tensor-product spaces
- This will allow us to develop the theory of the addition of angular momenta
- Finally, we will prove the Wigner-Eckart theorem, and discuss its consequences

Review of the basics

- The angular momentum operator J is Hermitian vector operator defined as

$$\mathrm{J} = J_x\hat{i} + J_y\hat{j} + J_z\hat{k}, \tag{1}$$

where J_x, J_y, and J_z are its three Cartesian components.

- The Hermiticity condition

$$\mathrm{J} = \mathrm{J}^\dagger, \tag{2}$$

implies that the individual components are also Hermitian

$$J_x = J_x^\dagger; \quad J_y = J_y^\dagger; \quad J_z = J_z^\dagger \tag{3}$$

Review of basics...

- Additionally, the three components of the angular momentum must satisfy the commutation relations

$$\begin{aligned} [J_x, J_y] &= i\hbar J_z \\ [J_y, J_z] &= i\hbar J_x \\ [J_z, J_x] &= i\hbar J_y \end{aligned} \tag{4}$$

- Which can be written in the compact form

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k \tag{5}$$

- Using these commutation relations, one can show that the operator $J^2 = J_x^2 + J_y^2 + J_z^2$, with each individual angular momentum component

$$\left[J^2, J_i\right] = 0 \tag{6}$$

Angular Momentum Algebra (contd.)

- Eq. 6 implies that J^2 and J_i are simultaneously diagonalizable, i.e., they have common eigenvectors
- But, because different components of J do not commute with each other (see Eq. 5), we cannot find their simultaneous eigenvectors
- Thus, by convention, we work with the simultaneous eigenvectors of J^2 and J_z, labeled $|jm\rangle$, satisfying

$$\begin{aligned} J^2|jm\rangle &= j(j+1)\hbar^2|jm\rangle \\ J_z|jm\rangle &= m\hbar|jm\rangle, \end{aligned} \tag{7}$$

  where $-j \leq m \leq j$.
- Kets $|jm\rangle$ form an orthonormal set

$$\langle j'm' \mid jm\rangle = \delta_{j'j}\delta_{m'm}. \tag{8}$$

Angular Momentum Algebra Revision...

- One can also show that the allowed values of j are

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \tag{9}$$

  and that successive m for a given j differ by one, i.e., $m' = m \pm 1$
- Combining this with that fact that $-j \leq m \leq j$, we conclude that for a given value of j, there are $2j+1$ allowed values of m, given by

$$-j, -j+1, -j+2, \ldots, j-2, j-1, j$$

- Now, the question arises, what is the action of J_x and J_y operators on the ket $|jm\rangle$?
- To perform these calculations, it helps to define the ladder operators

$$J_{\pm} = J_x \pm iJ_y. \tag{10}$$

Angular momentum algebra revision...

- It is easy to verify that the ladder operators are not Hermitian

$$J_{\pm}^{\dagger} = J_{\mp}. \tag{11}$$

- One can write J^2 operator in terms of them

$$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2. \tag{12}$$

- Using the commutation relations (Eq. 5) one can show that

$$J_+|jm\rangle = \sqrt{(j-m)(j+m+1)}\hbar\,|jm+1\rangle$$
$$J_-|jm\rangle = \sqrt{(j+m)(j-m+1)}\hbar|jm-1\rangle.$$

- Or in short

$$J_{\pm}|jm\rangle = \sqrt{(j\mp m)(j\pm m+1)}\hbar\,|jm\pm 1\rangle \tag{13}$$

Revision of Angular Momentum Algebra

- In other words the action of J_+/J_- on kets $|jm\rangle$ leaves j unchanged, but increments/decrements the m values by one.
- Using Eqs. 13 and 10, one can easily obtain the action of J_x/J_y operators on the ket $|jm\rangle$
- It is fruitful to make the following comment at this stage
- J_is refer to the Cartesian components of a general angular momentum operator
- In practice, J_i could be the orbital angular momentum operator L_i, or the spin angular momentum operator S_i, or the sum $L_i + S_i$ of the two.
- Or it could refer to an entirely different kind of angular momentum
- Any operator which satisfies the commutation relations of Eq. 5, will have the properties of a quantum angular momentum operator

Generator of Rotation

- Let us consider a general vector V, which could represent any vectorial physical quantity such as position r, momentum p etc.
- We will be interested in studying how a given vector transforms under a rotation
- For rotations, we can adopt a "passive" view or an "active" view.
- Under the "passive" view, the coordinate system (i.e. the coordinate axes) are rotated, keeping the vector fixed, and then we study how the vector transforms as a result
- In the "active" view, on the other hand, we hold the coordinate system fixed, and rotate the vector instead, and study its transformation properties

Generator of Rotation...

- Let us consider a system with Hamiltonian H
- We rotate the position vector r by an angle ϕ about an axis oriented along the direction $\hat{n}$.
- That is, we are adopting an active view of rotations.
- Let R denote the operator representing this rotation, under which $\mathrm{r} \to \mathrm{r}'$

$$\mathrm{r}' = R\mathrm{r}$$

- As a result, in the r-representation, the Hamiltonian operator $H(\mathrm{r})$, as well as a general wave function $\alpha(\mathrm{r})$ also transform

$$\begin{aligned} H(\mathrm{r}) &\to H'(\mathrm{r}') \\ \alpha(\mathrm{r}) &\to \alpha'(\mathrm{r}') \end{aligned} \tag{14}$$

- One can define these transformations using a unitary operator corresponding to the rotation R

Generator of Rotation...

- In the state space (Dirac representation), the corresponding unitary operator is denoted as U_R
- Its action on the Hamiltonian H and a general ket $|\alpha\rangle$ is given by

$$\begin{aligned} H_R &= U_R H U_R^\dagger \\ |\alpha\rangle_R &= U_R|\alpha\rangle. \end{aligned} \tag{15}$$

- Note that H_R and $|\alpha\rangle_R$ are the corresponding transformed quantities after the rotation R has been performed.
- Eqs. 15 are the state space counterparts of Eqs. 14.
- One can show that the unitary operator U_R is given by

$$U_R = e^{-\frac{i}{\hbar}\mathrm{J}\cdot\hat{n}\phi}, \tag{16}$$

where J is the vector angular momentum operator defined earlier.

Representation of the Rotation Operator

- Because J appears in the formula of the rotation operator, it is called the generator of rotations
- This is similar to the unitary operator $U(\mathrm{r})$ which defines a translation by a vector r

$$U(\mathrm{r}) = e^{-\frac{i}{\hbar}\mathrm{p}\cdot\mathrm{r}},$$

where p is the linear momentum operator.
- Thus, p is said to be the generator of translations.
- We know from linear algebra that the matrix corresponding to a linear operator in a vector space, with respect to a chosen basis, is called its representation
- We are interested in obtaining the representation of the U_R operator in the state space, with respect to the basis $\{|jm\rangle, m = -j, \ldots j\}$

Rotation Matrices...

- The matrices representing U_R with respect to the chosen basis are called rotation matrices
- Let us obtain the expressions for the elements of the rotation matrices
- Using the resolution of identity ($\sum_{m'=-j}^{j} |jm'\rangle \langle jm'| = I$), we obtain

$$U_R|jm\rangle = \sum_{m'=-j}^{j} |jm'\rangle \langle jm'|U_R|jm\rangle.$$

- Defining the rotation matrix elements as

$$\begin{aligned} D^{(j)}_{m'm}(R) &= \langle jm'|U_R|jm\rangle \\ &= \left\langle jm' \left| e^{-\frac{i}{\hbar}\mathbf{J}\cdot\hat{n}\phi} \right| jm \right\rangle, \end{aligned} \tag{17}$$

we obtain

$$U_R|jm\rangle = \sum_{m'=-j}^{j} D^{(j)}_{m'm}(R)|jm'\rangle. \tag{18}$$

Rotation matrices...

- From Eq. 17 it is obvious that the elements $D^{(j)}_{m'm}(R)$ define the representation of the rotation operator with respect to the chosen basis, i.e., the rotation matrices.
- It is obvious that computing $D^{(j)}_{m'm}(R)$ for the most general rotation will be complicated
- However, for a rotation about the z axis ($\hat{n} = \hat{k}$), the matrix elements have a very simple form, as derived below

$$
\begin{aligned}
D^{(j)}_{m'm}(R) &= \left\langle jm' \middle| e^{-\frac{i}{\hbar}\mathbf{J}\cdot\hat{n}\phi} \middle| jm \right\rangle \\
&= \left\langle jm' \middle| e^{-\frac{i}{\hbar}J_z\phi} \middle| jm \right\rangle \\
&= \left\langle jm' \middle| e^{-\frac{i}{\hbar}m\hbar\phi} \middle| jm \right\rangle \\
&= e^{-im\phi} \left\langle jm' | jm \right\rangle \\
&= e^{-im\phi} \delta_{m'm}.
\end{aligned}
$$

Rotation matrices through Euler Angles

- In the derivation we used the relation

$$f(J_z)|jm\rangle = f(m\hbar)|jm\rangle,$$

  where $f(J_z)$ is an analytic function of J_z.
- The result can be easily proved by making a Taylor expansion of $f(J_z)$, and the fact $J_z|jm\rangle = m\hbar|jm\rangle$.
- Rotations about a general axis can be simplified a great deal by borrowing the concept of Euler angles from rigid-body dynamics
- Using the concept of Euler angles or Euler rotations, a general rotation can be expressed in terms of three counter-clockwise rotations by angles α, β, and γ (called Euler angles)
- The first rotation by angle α is about the original z axis
- The second one by angle β is about the new y axis
- The final one by angle γ is about the new z axis.

Rotation matrices using Euler angles...

- Note that here we are rotating the coordinate system, which means these are "passive" rotations
- If the initial axes are defined as (x, y, z), intermediate ones by (x'', y'', z''), and the final ones by (x', y', z'), then it is obvious

$$U_R = e^{-\frac{i}{\hbar}\gamma\hat{z}'\cdot\mathbf{J}} e^{-\frac{i}{\hbar}\beta\hat{y}''\cdot\mathbf{J}} e^{-\frac{i}{\hbar}\alpha\hat{z}\cdot\mathbf{J}}. \tag{19}$$

- Using the following mathematical trick, one can transform U_R into a form which involves rotations only about the original (unprimed) axes.
- The involves the realization

$$e^{-\frac{i}{\hbar}\beta\hat{y}''\cdot\mathbf{J}} = e^{-\frac{i}{\hbar}\alpha\hat{z}\cdot\mathbf{J}} e^{-\frac{i}{\hbar}\beta\hat{y}\cdot\mathbf{J}} e^{\frac{i}{\hbar}\alpha\hat{z}\cdot\mathbf{J}}, \tag{20}$$

Rotation matrixes using Euler angles...

- and

$$e^{-\frac{i}{\hbar}\gamma\hat{z}'\cdot\mathbf{J}} = e^{-\frac{i}{\hbar}\beta\hat{y}''\cdot\mathbf{J}} e^{-\frac{i}{\hbar}\gamma\hat{z}\cdot\mathbf{J}} e^{\frac{i}{\hbar}\beta\hat{y}''\cdot\mathbf{J}}. \tag{21}$$

- On substituting Eqs. 20 and 21 in Eq. 19, we obtain the desired expression

$$\begin{aligned} U_R &= e^{-\frac{i}{\hbar}\alpha\hat{z}\cdot\mathbf{J}} e^{-\frac{i}{\hbar}\beta\hat{y}\cdot\mathbf{J}} e^{-\frac{i}{\hbar}\gamma\hat{z}\cdot\mathbf{J}} \\ &= e^{-\frac{i}{\hbar}\alpha J_z} e^{-\frac{i}{\hbar}\beta J_y} e^{-\frac{i}{\hbar}\gamma J_z} \end{aligned} \tag{22}$$

- This leads to a much simpler expression for a general rotation matrix

$$\begin{aligned} D^{(j)}_{m'm}(R) &= \left\langle jm' \left| e^{-\frac{i}{\hbar}\alpha J_z} e^{-\frac{i}{\hbar}\beta J_y} e^{-\frac{i}{\hbar}\gamma J_z} \right| jm \right\rangle \\ &= e^{-im'\alpha} e^{-im\gamma} \left\langle jm' \left| e^{-\frac{i}{\hbar}\beta J_y} \right| jm \right\rangle \\ &= e^{-im'\alpha} e^{-im\gamma} \left\langle jm' \left| e^{-\frac{\beta}{2\hbar}(J_+ - J_-)} \right| jm \right\rangle \end{aligned} \tag{23}$$

Rotation matrices

- One can also verify the following symmetry properties of the rotation matrices

$$
\begin{aligned}
D^{(j)\star}_{m'm}(\alpha,\beta,\gamma) &= D^{(j)}_{m'm}(-\gamma,-\beta,-\alpha) \\
D^{(j)^*}_{m'm}(\alpha,\beta,\gamma) &= (-1)^{m-m'} D^{(j)}_{-m',-m}(\alpha,\beta,\gamma).
\end{aligned} \tag{24}
$$

Orbital Angular Momentum and Rotation Matrices

- If $j = l$, where l is a non-negative integer, we have

$$\langle \mathrm{r} \mid lm \rangle = Y_{lm}(\theta, \phi), \tag{25}$$

- Above $Y_{lm}(\theta, \phi)$ is a spherical harmonic, an eigenfunction of the L^2 and L_z operators

$$\begin{aligned} L^2|lm\rangle &= l(l+1)\hbar^2|lm\rangle \\ L_z|lm\rangle &= m\hbar|lm\rangle \end{aligned} \tag{26}$$

- Let us explore the influence of a rotation R on spherical harmonics.

$$\begin{aligned} |lm\rangle' &= U_R|lm\rangle \\ &= \sum_{m'=-l}^{l} \langle lm'|U_R|lm\rangle|lm'\rangle \\ &= \sum_{m'=-l}^{l} D^{(l)}_{m'm}(R)\,|lm'\rangle \end{aligned} \tag{27}$$

Orbital angular momentum...

- On taking the projection of Eq. 27 in r space, we have

$$\langle \mathrm{r}|lm\rangle' = \sum_{m'=-l}^{l} D^{(l)}_{m'm}(R)\langle \mathrm{r}\left|lm'\right\rangle,$$

leading to

$$Y_{lm}(\theta', \phi') = \sum_{m'=-l}^{l} D^{(l)}_{m'm}(R) Y_{lm'}(\theta, \phi), \tag{28}$$

where (θ, ϕ) and (θ', ϕ') denote the coordinates of the same point in space, but with respect to the initial and the rotated coordinate axes.

Orbital angular momentum...

- Using the unitary property of the rotation matrices, one can easily invert Eq. 28 above to obtain

$$Y_{lm}(\theta,\phi) = \sum_{m'=-l}^{l} D^{(l)*}_{mm'}(R) Y_{lm'}(\theta',\phi'), \tag{29}$$

- For a point on the z' axis, $\theta' = 0$, while for the same point $\theta = \beta$ and $\phi = \alpha$. Using this, and the fact that

$$Y_{lm'}(\theta' = 0, \phi') = \sqrt{\frac{2l+1}{4\pi}}\delta_{m'0}, \tag{30}$$

we obtain from Eq. 29

$$Y_{lm}(\beta,\alpha) = \sum_{m'=-l}^{l} D^{(l)*}_{mm'}(R)\sqrt{\tfrac{2l+1}{4\pi}}\delta_{m'0} \quad ,$$

leading to

$$D^{(l)*}_{m0}(R) = \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\beta,\alpha). \tag{31}$$

Orbital Angular Momentum...

- Using Eqs. 28 and 31, we can prove another interesting result
- For the purpose, we set $m = 0$ in Eq. 28

$$Y_{l0}(\theta',\phi') = \sum_{m'=-l}^{l} D^{(l)}_{m'0}(R)\,Y_{lm'}(\theta,\phi)$$
$$= \sqrt{\frac{4\pi}{2l+1}} \sum_{m'=-l}^{l} Y^{*}_{lm'}(\beta,\alpha)\,Y_{lm'}(\theta,\phi).$$

- Using the fact that $Y_{l0}(\theta',\phi') = \sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta')$, we obtain from above

$$P_l\left(\cos\theta'\right) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^{*}_{lm}(\beta,\alpha)\,Y_{lm}(\theta,\phi), \qquad (32)$$

which is a very useful mathematical result called “addition theorem of spherical harmonics”.

Direct Sum and Direct Product Spaces

- Suppose we have two vector spaces $\mathscr{E}_1$ with basis $\{|a_i\rangle, i = 1, \ldots, n\}$ and $\mathscr{E}_2$ with basis $\{|b_j\rangle, j = 1, \ldots, m\}$
- Using the operations of direct sum and direct product one can construct larger dimensional spaces, as compared to the original spaces, as explained below.
- **Direct Sum:** The direct sum space of $\mathscr{E}$ of $\mathscr{E}_1$ and $\mathscr{E}_2$ is defined as

$$\mathscr{E} = \mathscr{E}_1 \oplus \mathscr{E}_2, \tag{33}$$

  above $\oplus$ sign indicates the operation of direct sum. The vector space $\mathscr{E}$ has dimension $n + m$, with the ordered basis $\{|a_1\rangle, |a_2\rangle, \ldots |a_n\rangle, |b_1\rangle, |b_2\rangle, \ldots |b_m\rangle\}$

Direct sum...

- Next, we demonstrate the operation of direct sum for the case of two vectors
- Let us consider two kets $|v\rangle \in \mathscr{E}_1$ and $|u\rangle \in \mathscr{E}_2$, so that

$$\begin{aligned} |v\rangle &= \sum_{i=1}^{n} v_i |a_i\rangle \\ |u\rangle &= \sum_{j=1}^{m} u_j |b_j\rangle, \end{aligned} \tag{34}$$

- These two kets can be expressed as column vectors

$$\begin{aligned} |v\rangle &\equiv \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\ |u\rangle &\equiv \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \end{aligned} \tag{35}$$

Direct sum...

- Then the direct sum of the two vectors $w = v \oplus u$, can be represented as the column vector

$$w \equiv \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ u_1 \\ \vdots \\ u_m \end{pmatrix}. \tag{36}$$

- For linear operators $A : \mathscr{E}_1 \to \mathscr{E}_1$ and $B : \mathscr{E}_2 \to \mathscr{E}_2$, represented as

$$A \equiv \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \vdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \tag{37}$$

Direct sum...

- and

$$B \equiv \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \vdots & \vdots \\ b_{m1} & \cdots & b_{mm} \end{pmatrix}, \tag{38}$$

- the representation of $C = A \oplus B$, where $C : \mathscr{E} \to \mathscr{E}$ will be

$$C \equiv \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{11} & \cdots & b_{1m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & b_{m1} & \cdots & b_{mm} \end{pmatrix} \tag{39}$$

Direct Sum...

- In a shorthand notation, one can write Eq. 39 as

$$C \equiv \begin{pmatrix} A & O \\ O^T & B \end{pmatrix}, \tag{40}$$

where O denotes a $n \times m$ dimensional null matrix.

Direct/Tensor Product

- The direct (or tensor) product space of $\mathscr{E}_1$ and $\mathscr{E}_2$ is denoted as

$$\mathscr{E} = \mathscr{E}_1 \otimes \mathscr{E}_2, \tag{41}$$

  and is an nm dimensional space with the ordered basis $\{|a_i\rangle \otimes |b_j\rangle, i = 1, \ldots, n; j = 1, \ldots, m\}$.
- Various notations used to depict the direct product basis are

$$|a_i\rangle \otimes |b_j\rangle = |a_i\rangle|b_j\rangle = |a_i b_j\rangle. \tag{42}$$

- Similar to direct sum, one can have the direct product of two vectors, as well as two operators belonging to state spaces $\mathscr{E}_1$ and $\mathscr{E}_2$.
- Let us consider two kets $|v\rangle \in \mathscr{E}_1$ and $|u\rangle \in \mathscr{E}_2$, defined in Eqs. 34 and 35.

Direct product of vectors

- The direct product of these two kets $|w\rangle = |v\rangle \otimes |u\rangle$ is defined as

$$|w\rangle = |v\rangle \otimes |u\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} v_i u_j |a_i b_j\rangle.$$

- For the chosen ordered basis, the representation of $|w\rangle$ is

$$|w\rangle \equiv \begin{pmatrix} v_1 u_1 \\ \vdots \\ v_1 u_m \\ v_2 u_1 \\ \vdots \\ v_2 u_m \\ \vdots \\ v_n u_1 \\ \vdots \\ v_n u_m \end{pmatrix} \tag{43}$$

Direct product of operators

- Let us again consider linear operators $A : \mathscr{E}_1 \to \mathscr{E}_1$ and $B : \mathscr{E}_2 \to \mathscr{E}_2$, whose representations with respect to the given ordered basis are given by Eqs. 37 and 38.
- The matrix elements of A and B are given by $A_{ij} = \langle a_i | A | a_j \rangle$ and $B_{kl} = \langle b_k | B | b_l \rangle$
- Let us compute the matrix elements of the operator $C : \mathscr{E} \to \mathscr{E}$ which is the direct product of A and B

$$C = A \otimes B.$$

- So that

$$\begin{aligned} C_{ik;jl} &= \langle a_i b_k | A \otimes B | a_j b_l \rangle \\ &= \langle a_i | A | a_j \rangle \langle b_k | B | b_l \rangle \\ &= A_{ij} B_{kl}. \end{aligned} \quad (44)$$

Direct product of operators

- Assuming the ordered basis to be the same as considered earlier for direct product of kets $\{|a_1 b_1\rangle, \ldots, |a_1 b_m\rangle, |a_2 b_1\rangle, \ldots, |a_2 b_m\rangle, \ldots, |a_n b_1\rangle, \ldots, |a_n b_m\rangle\}$
- We obtain the following matrix representation of C

$$C = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \vdots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}, \tag{45}$$

where $a_{ij}B$ is the matrix obtained by multiplying each element of the B matrix by a_{ij}

$$a_{ij}B = \begin{pmatrix} a_{ij}b_{11} & \cdots & a_{ij}b_{1m} \\ \vdots & \vdots & \vdots \\ a_{ij}b_{m1} & \cdots & a_{ij}b_{mm} \end{pmatrix}. \tag{46}$$

An Example of a Direct Product of Matrices

- Let us illustrate the procedure of computing the direct product by considering an example involving 2×2 matrices.
- Let $n = m = 2$, so that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

- So that $C = A \otimes B$ is given by

$$C = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}.$$

Addition of Angular Momenta

- Suppose we have two distinct angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$, which may belong to two different particles of a given system
- Or may correspond to two different types of angular momenta (say $\mathbf{L}$ and $\mathbf{S}$) of the same particle
- Let $\mathbf{J}_1$ and $\mathbf{J}_2$ belong to state spaces $\mathscr{E}_1$ and $\mathscr{E}_2$
- Then the total angular momentum $\mathbf{J}$ obtained by adding $\mathbf{J}_1$ and $\mathbf{J}_2$ will be symbolically denoted as $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$
- But, as we know that $\mathscr{E}_1$ and $\mathscr{E}_2$ are different spaces with different dimensions, in general.
- Therefore, we cannot simply add quantities belonging to different state spaces

Addition of Angular Momenta...

- As a matter of fact J belongs to the direct product space $\mathscr{E} = \mathscr{E}_1 \otimes \mathscr{E}_2$.
- We will show that the mathematically rigorous manner of adding the two angular momenta is

$$\mathrm{J} = \mathrm{J}_1 \otimes I_2 + I_1 \otimes \mathrm{J}_2, \tag{47}$$

  where $I_1 \in \mathscr{E}_1$ and $I_2 \in \mathscr{E}_2$ are the identity operators.
- Let us consider a rotation by an angle ϕ about an axis oriented along the direction $\hat{n}$.
- Because J is the angular momentum operator in the direct product space $\mathscr{E} = \mathscr{E}_1 \otimes \mathscr{E}_2$, therefore it must generate rotations in that space

$$U_R^{(\mathscr{E})} = e^{-\frac{i}{\hbar}\mathrm{J}\cdot\hat{n}\phi}. \tag{48}$$

Addition of angular momenta...

- Similarly, $\mathbf{J}_1$ and $\mathbf{J}_2$ are generators of rotations in spaces $\mathscr{E}_1$ and $\mathscr{E}_2$, as a result of which

$$U_R^{(\mathscr{E}_1)} = e^{-\frac{i}{\hbar}\mathbf{J}_1\cdot\hat{n}\phi}$$
$$U_R^{(\mathscr{E}_2)} = e^{-\frac{i}{\hbar}\mathbf{J}_2\cdot\hat{n}\phi}. \tag{49}$$

- Because, $\mathscr{E} = \mathscr{E}_1 \otimes \mathscr{E}_2$, therefore

$$U_R^{(\mathscr{E})} = U_R^{(\mathscr{E}_1)} \otimes U_R^{(\mathscr{E}_2)}, \tag{50}$$

which implies that

$$e^{-\frac{i}{\hbar}\mathbf{J}\cdot\hat{n}\phi} = e^{-\frac{i}{\hbar}\mathbf{J}_1\cdot\hat{n}\phi} \otimes e^{-\frac{i}{\hbar}\mathbf{J}_2\cdot\hat{n}\phi} \tag{51}$$

Addition of angular momenta...

- The RHS of the previous equation (Eq. 51) can be rewritten as

$$e^{-\frac{i}{\hbar}\mathsf{J}_1\cdot\hat{n}\phi}\otimes e^{-\frac{i}{\hbar}\mathsf{J}_2\cdot\hat{n}\phi} = (e^{-\frac{i}{\hbar}(\mathsf{J}_1\otimes I_2)\cdot\hat{n}\phi})(e^{-\frac{i}{\hbar}(I_1\otimes\mathsf{J}_2)\cdot\hat{n}\phi})$$
$$= e^{-\frac{i}{\hbar}(\mathsf{J}_1\otimes 1_2 + I_1\otimes\mathsf{J}_2)\cdot\hat{n}\phi}$$

- The last step above was possible because J_1 and J_2 commute with each other, as they are in different spaces. This, on substitution in Eq. 51 leads to

$$e^{-\frac{i}{\hbar}\mathsf{J}\cdot\hat{n}\phi} = e^{-\frac{i}{\hbar}(\mathsf{J}_1\otimes I_2 + I_1\otimes\mathsf{J}_2)\cdot\hat{n}\phi}. \tag{52}$$

- On comparing the two sides, we obtain the desired result of Eq. 47

$$\mathsf{J} = \mathsf{J}_1\otimes I_2 + I_1\otimes\mathsf{J}_2.$$

- As mentioned earlier, this result is often written in an informal manner as a simple addition of J_1 and J_2 operators

$$\mathsf{J} = \mathsf{J}_1 + \mathsf{J}_2. \tag{53}$$

- Because $[\mathrm{J}_1, \mathrm{J}_2] = 0$, therefore, it is easy to prove that J^2 and various components J_i satisfy the same commutation relations satisfied by J_1^2, J_{1i}, J_2^2, and J_{2i}

$$\begin{aligned} \left[J^2, J_i\right] &= 0 \\ \left[J_i, J_j\right] &= i\hbar\varepsilon_{ijk} J_k \end{aligned} \tag{54}$$

- Eq. 54 implies that there exists a basis $|jm\rangle$ which are the common eigenvectors of J^2 and J_z

$$\begin{aligned} J^2|jm\rangle &= \hbar^2 j(j+1)\,|jm\rangle \\ J_z\,|jm\rangle &= m\hbar\,|jm\rangle \end{aligned} \tag{55}$$

- But it is easy to see

$$[J_1^2, J_z] = [J_2^2, J_z] = [J^2, J_1^2] = [J^2, J_2^2] = 0 \tag{56}$$

- This means that $|jm\rangle$ states must also be eigenvectors of J_1^2 and J_2^2 operators, in addition J^2, and J_z

$$\begin{aligned} J_1^2|jm\rangle &= j_1(j_1+1)\hbar^2\,|jm\rangle \\ J_2^2|jm\rangle &= j_2(j_2+1)\hbar^2\,|jm\rangle\,. \end{aligned} \tag{57}$$

- Therefore, we adopt the notation

$$|\,jm\rangle \rightarrow |j_1 j_2 jm\rangle, \tag{58}$$

which indicates that these states are eigenvectors of operators of J_1^2, J_2^2, J^2, and J_z.
- Clearly, states $|j_1 j_2 jm\rangle \in \mathscr{E}$, which is a direct product space
- But, direct product states $|j_1 j_2 m_1 m_2\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle$ also belong to $\mathscr{E}$
- Therefore, the two sets of states must be related to each other by a unitary transformation, because both form orthonormal sets, and span the same state space $\mathscr{E}$

$$\begin{aligned} \langle j_1' j_2' j' m' | j_1 j_2 jm\rangle &= \delta_{j_1' j_1}\,\delta_{j_2' j_2}\,\delta_{j' j}\,\delta_{m' m} \\ \langle j_1' j_2' m_1' m_2' | j_1 j_2 m_1 m_2\rangle &= \delta_{j_1' j_1}\,\delta_{j_2' j_2}\,\delta_{m_1' m_1}\,\delta_{m_2' m_2} \end{aligned} \tag{59}$$

Clebsch-Gordon Coefficients

- To establish the connection between the states $|j_1 j_2 j m\rangle$ and $|j_1 j_2 m_1 m_2\rangle$, we make use of the resolution of identity for fixed values of j_1 and j_2

$$I = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2|, \qquad (60)$$

and apply it on the state $|j_1 j_2 j m\rangle$

$$\begin{aligned} |j_1 j_2 j m\rangle &= I|j_1 j_2 j m\rangle \\ &= \sum_{m_1, m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2|j_1 j_2 j m\rangle. \end{aligned} \qquad (61)$$

- From Eq. 61, it is obvious that the two sets of states are connected by expansion coefficients $\langle j_1 j_2 m_1 m_2|j_1 j_2 j m\rangle$, called Clebsch-Gordon coefficients
- Next, we will study their properties, and develop approaches for computing them.

Clebsch-Gordon Coefficients...

- Let us apply $J_z = J_{1z} + J_{2z}$ operator on both the sides of Eq. 61

$$J_z|j_1 j_2 j m\rangle = \sum_{m_1, m_2} (J_{1z} + J_{2z})|j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2|j_1 j_2 j m\rangle$$

- This leads to

$$m\hbar|j_1 j_2 j m\rangle = \sum_{m_1, m_2} (m_1 + m_2)\hbar|j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2|j_1 j_2 j m\rangle.$$

- Taking the inner product of this equation with $|j_1 j_2 m_1' m_2'\rangle$ on both the sides, and making use of the orthonormality relations (Eq. 59) we obtain

$$m\langle j_1 j_2 m_1' m_2'|j_1 j_2 j m\rangle = (m_1' + m_2')\langle j_1 j_2 m_1' m_2'|j_1 j_2 j m\rangle$$
$$\implies m\langle j_1 j_2 m_1 m_2|j_1 j_2 j m\rangle = (m_1 + m_2)\langle j_1 j_2 m_1 m_2|j_1 j_2 j m\rangle$$

- Leading to

$$(m - m_1 - m_2)\,\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle = 0.$$

- Clearly $\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle \neq 0$, only if

$$m = m_1 + m_2. \tag{62}$$

- This formula is called "conservation of m" or "m selection rule"
- This implies that only those Clebsch-Gordon coefficients (CGCs) $\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$ will be non-vanishing for which $m = m_1 + m_2$

Recursion Relations of Clebsch-Gordon Coefficients

- Next, we apply J_+ and J_- operators on Eq. 61 to derive important recursions relations involving CGCs.
- Note that here

$$J_\pm = J_{1\pm} + J_{2\pm}. \tag{63}$$

- With this

$$J_\pm|j_1 j_2 jm\rangle = \sum_{m_1,m_2} (J_{1\pm} + J_{2\pm})|j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2|j_1 j_2 jm\rangle. \tag{64}$$

- But, using Eqs. 13 , we obtain

$$J_\pm|j_1 j_2 jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\hbar \mid j_1 j_2 jm \pm 1\rangle$$
$$(J_{1\pm} + J_{2\pm})|j_1 j_2 m_1 m_2\rangle = \sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)}\hbar|j_1 j_2 m_1 \pm 1 m_2\rangle + \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)}\hbar|j_1 j_2 m_1 m_2 \pm 1\rangle \tag{65}$$

CGC Recursion Relations

- Substituting Eqs. 65 in Eq. 64, we obtain

$$\begin{aligned}&\sqrt{(j\mp m)(j\pm m+1)}\hbar\mid j_1j_2jm\pm 1\rangle\\ &=\sum_{m_1,m_2}\Big\{\sqrt{(j_1\mp m_1)(j_1\pm m_1+1)}\hbar|j_1j_2m_1\pm 1m_2\rangle\\ &+\sqrt{(j_2\mp m_2)(j_2\pm m_2+1)}\hbar|j_1j_2m_1m_2\pm 1\rangle\Big\}\langle j_1j_2m_1m_2|j_1j_2jm\rangle\end{aligned} \tag{66}$$

- We take the inner product of Eq. 66 with $|j_1j_2m'_1m'_2\rangle$, and use the orthonormality relations of Eq. 59 to obtain

$$\begin{aligned}&\sqrt{(j\mp m)(j\pm m+1)}\langle j_1j_2m'_1m'_2\mid j_1j_2jm\pm 1\rangle\\ &=\sum_{m_1,m_2}\Big\{\sqrt{(j_1\mp m_1)(j_1\pm m_1+1)}\delta_{m'_1m_1\pm 1}\delta_{m'_2m_2}\\ &+\sqrt{(j_2\mp m_2)(j_2\pm m_2+1)}\delta_{m'_1m_1}\delta_{m'_2m_2\pm 1}\Big\}\langle j_1j_2m_1m_2|j_1j_2jm\rangle,\end{aligned}$$

which leads to

- $$\begin{aligned}&\sqrt{(j\mp m)(j\pm m+1)}\langle j_1j_2m'_1m'_2\mid j_1j_2jm\pm 1\rangle\\ &=\sqrt{(j_1\mp m'_1+1))(j_1\pm m'_1)}\langle j_1j_2m'_1\mp 1m_2|j_1j_2jm\rangle\\ &+\sqrt{(j_2\mp m'_2+1))(j_2\pm m'_2)}\langle j_1j_2m_1m'_2\mp 1|j_1j_2jm\rangle\end{aligned}$$

- Next, we just replace m_1' and m_2' by m_1 and m_2, respectively, to obtain the final expression for the desired recursion relation

$$
\begin{aligned}
&\sqrt{(j \mp m)(j \pm m + 1)}\, \langle j_1 j_2 m_1 m_2 \mid j_1 j_2 j m \pm 1 \rangle \\
&= \sqrt{(j_1 \pm m_1)(j_1 \mp m_1 + 1)}\, \langle j_1 j_2 m_1 \mp 1 m_2 \mid j_1 j_2 j m \rangle \\
&+ \sqrt{(j_2 \pm m_2)(j_2 \mp m_2 + 1)}\, \langle j_1 j_2 m_1 m_2 \mp 1 \mid j_1 j_2 j m \rangle
\end{aligned}
\tag{67}
$$

Calculating CGCs using Recursion Relations

- Using recursion relation of Eq. 67, one can compute, for a given set of j_1, j_2, and j, all non-vanishing CGCs, in terms of just one of them.
- Let us choose maximum allowed values for m_1 and m: $m_1 = j_1,\ m = j$
- And let $m_2 = j - j_1 - 1$
- Substituting these in Eq. 67 with the lower sign, we obtain

$$\sqrt{(j+j)(j-j+1)}\,\langle j_1 j_2 j_1 j-j_1-1 \mid j_1 j_2 j j-1\rangle$$
$$= \sqrt{(j_1-j_1)(j_1+j_1+1)}\,\langle j_1 j_2 j_1+1 j-j_1-1 \mid j_1 j_2 j j\rangle$$
$$+\sqrt{(j_2-(j-j_1-1))\,(j_2+j-j_1-1+1)}\,\langle j_1 j_2 j_1 j-j_1-1+1 \mid j_1 j_2 j j\rangle$$

Calculations of CGCs...

- We note that the first term on the RHS of the previous equation vanishes, as a result of which we obtain

$$\begin{aligned}&\langle j_1 j_2 j_1 j - j_1 - 1 \mid j_1 j_2 j j - 1\rangle \\ &= \sqrt{\tfrac{(j_1+j_2+1-j)(j+j_2-j_1)}{2j}}\,\langle j_1 j_2 j_1 j - j_1 \mid j_1 j_2 j j\rangle\end{aligned} \tag{68}$$

- From Eq. 68 we can compute the CGC $\langle j_1 j_2 j_1 j - j_1 - 1 \mid j_1 j_2 j j - 1\rangle$, provided the value of $\langle j_1 j_2 j_1 j - j_1 \mid j_1 j_2 j j\rangle$ is known
- Let us again use the recursion relations of Eq. 67, but using the upper sign, and $m_1 = j_1$, $m = j - 1$, and $m_2 = j - j_1$

$$\begin{aligned}&\sqrt{(j-(j-1))(j+(j-1)+1)}\,\langle j_1 j_2 j_1 j - j_1 \mid j_1 j_2 j (j-1)+1\rangle \\ &= \sqrt{(j_1+j_1)(j_1-j_1+1)}\,\langle j_1 j_2 j_1 - 1 j - j_1 \mid j_1 j_2 j j - 1\rangle \\ &+ \sqrt{(j_2+j-j_1)(j_2-(j-j_1)+1)}\,\langle j_1 j_2 j_1 j - j_1 - 1 \mid j_1 j_2 j j - 1\rangle\end{aligned}$$

- This simplifies to

$$\sqrt{2j}\,\langle j_1 j_2 j_1 j - j_1 \mid j_1 j_2 j j\rangle = \sqrt{2j_1}\,\langle j_1 j_2 j_1 - 1 j - j_1 \mid j_1 j_2 j j - 1\rangle + \sqrt{(j_2 + j - j_1)(j_1 + j_2 - j + 1)}\,\langle j_1 j_2 j_1 j - j_1 - 1 \mid j_1 j_2 j j - 1\rangle\,,$$

  leading to the final form

$$\langle j_1 j_2 j_1 - 1 j - j_1 \mid j_1 j_2 j - 1\rangle = \sqrt{\frac{j}{j_1}}\,\langle j_1 j_2 j_1 j - j_1 \mid j_1 j_2 j j\rangle - \sqrt{\frac{(j_2 + j - j_1)(j_1 + j_2 - j + 1)}{2j_1}}\,\langle j_1 j_2 j_1 j - j_1 - 1 \mid j_1 j_2 j j - 1\rangle\,. \quad (69)$$

- Thus Eq. 69 allows us to compute the CGC $\langle j_1 j_2 j_1 - 1 j - j_1 \mid j_1 j_2 j - 1\rangle$ if we know the values of $\langle j_1 j_2 j_1 j - j_1 \mid j_1 j_2 j j\rangle$ and $\langle j_1 j_2 j_1 j - j_1 - 1 \mid j_1 j_2 j j - 1\rangle$.
- Thus, using the recursion relations (Eq. 67) we can compute all the CGCs, provided we know the value of $\langle j_1 j_2 j_1 j - j_1 \mid j_1 j_2 j j\rangle$.

Triangular Inequality of CGCs

- Let us derive another important selection rule for CGCs, which allows us to compute all allowed values of j, for the given values of j_1 and j_2.
- Let us consider the CGC $\langle j_1 j_2 j_1 j - j_1 \mid j_1 j_2 j j \rangle$
- Because $j - j_1$ is a possible value of m_2, therefore, it must satisfy

  $$-j_2 \leq j - j_1 \leq j_2,$$

  from which we obtain

  $$j_1 - j_2 \leq j \leq j_1 + j_2 \tag{70}$$

- Similarly, if we consider the CGC $\langle j_1 j_2 j - j_2 j_2 \mid j_1 j_2 j j \rangle$, we have

  $$-j_1 \leq j - j_2 \leq j_1,$$

  leading to

  $$j_2 - j_1 \leq j \leq j_1 + j_2. \tag{71}$$

- We can combine the results of Eqs. 70 and 71 in a single inequality
$$|j_1 - j_2| \leq j \leq j_1 + j_2, \tag{72}$$
which is the famous triangular inequality.
- Triangular inequality is nothing but a selection rule for CGCs, in addition to the "m selection rule" derived earlier.
- If for a given pair of values of j_1 and j_2, j does not satisfy triangular inequality, the corresponding CGC will surely vanish.

Orthonormality Conditions of CGCs

- Next we derive two orthonormality conditions satisfied by the CGCs
- They are quite important, although both of them can be derived quite easily using the "resolution of identity".
- We know from Eq. 59 that

$$\langle j_1 j_2 j' m' | j_1 j_2 j m \rangle = \delta_{j'j} \delta_{m'm}.$$

- We apply the resolution of identity $\sum_{m_1,m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2| = I$ on the left hand side of the equation above to obtain

$$\sum_{m_1,m_2} \langle j_1 j_2 j' m' | j_1 j_2 m_1 m_2 \rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle = \delta_{j'j} \delta_{m'm}$$

- Assuming that the CGCs are real, i.e.,

$$\langle j_1 j_2 j' m' | j_1 j_2 m_1 m_2 \rangle = \langle j_1 j_2 m_1 m_2 | j_1 j_2 j' m' \rangle$$

Orthonormality of CGCs...

- Using this in the previous equation, we obtain the first orthonormality relation of the CGCs

$$\sum_{m_1,m_2} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j' m' \rangle = \delta_{j' j} \delta_{m' m}. \quad (73)$$

- Next, we derive the second orthonormality relation starting with (Eq. 59)

$$\langle j_1 j_2 m_1' m_2' | j_1 j_2 m_1 m_2 \rangle = \delta_{m_1' m_1} \delta_{m_2' m_2}$$

- We insert the resolution of identity $\sum_{j,m} |j_1 j_2 j m\rangle \langle j_1 j_2 j m| = I$ on the left to obtain

$$\sum_{j,m} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m \rangle \langle j_1 j_2 j m | j_1 j_2 m_1 m_2 \rangle = \delta_{m_1' m_1} \delta_{m_2' m_2}.$$

- Again using the reality of CGCs, we obtain our final orthonormality relation for CGCs

$$\sum_{j,m} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m \rangle \langle j_2 m_1 m_2 | j j_2 j m \rangle = \delta_{m_1' m_1} \delta_{m_2' m_2}. \quad (74)$$

Orthonormality of CGCs...

- From Eqs. 73 and 74 it is obvious that CGCs form a unitary matrix
- Orthonormality condition of Eq. 73 and recursion relations (Eq. 67) are used to compute the CGCs $\langle j_1 j_2 j_1 j - j_1 \mid j_1 j_2 j j \rangle$ which by convention are assumed to be real and positive.
- Rest of the CGCs can be obtained by further applications of the recursion relations, as will be demonstrated in the tutorial problems.

Clebsch-Gordon Series

- From the previous discussion it is easy to deduce that the number of direct product (or uncoupled) basis states $|j_1 j_2 m_1 m_2\rangle \in \mathscr{E} = \mathscr{E}_1 \otimes \mathscr{E}_2$ is identical to the number of coupled states $|j_1 j_2 jm\rangle$, which also belong to $\mathscr{E}$
- The number of direct product basis states is easy to count $(2j_1+1)(2j_2+1)$. You will get the same number in the coupled representation also (try it yourself).
- This is because the two sets of states are connected by a unitary transformation whose matrix elements are CGCs
- There are several consequences of this, which we explore next
- The resolution of identity in the two basis sets must be identical, which means

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2| = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} |j_1 j_2 jm\rangle\langle j_1 j_2 jm|. \tag{75}$$

- If we examine the RHS of Eq. 75 carefully, we realize that each one of the kets $|j_1 j_2 j m\rangle$ belongs to a $2j+1$ dimensional space.
- And different values of j correspond to a different space, which is a subspace of $\mathscr{E}$.
- As a result, the sum on the RHS of Eq. 75 is actually a direct sum, i.e.,

$$\begin{aligned}
&\sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} |j_1 j_2 j m\rangle\langle j_1 j_2 j m| \\
&= \sum_{m=-|j_1-j_2|}^{|j_1-j_2|} |j_1 j_2 |j_1 - j_2| m\rangle\langle j_1 j_2 |j_1 - j_2| m| \\
&\oplus \sum_{m=-|j_1-j_2|-1}^{|j_1-j_2|+1} |j_1 j_2 |j_1 - j_2| + 1 m\rangle\langle j_1 j_2 |j_1 - j_2| + 1 m| \\
&\oplus \cdots \oplus \sum_{m=-(j_1+j_2)}^{j_1+j_2} |j_1 j_2 j_1 + j_2 m\rangle\langle j_1 j_2 j_1 + j_2 m|
\end{aligned} \tag{76}$$

Clebsch-Gordon Series...

- Each term in the direct sum of Eq. 76 corresponds to resolution of identity in that subspace
- If we adopt the notation

$$\sum_{m=-j}^{j} |j_1 j_2 jm\rangle\langle j_1 j_2 jm| = I_j, \tag{77}$$

- then the Eq. 76 can be written as

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} |j_1 j_2 jm\rangle\langle j_1 j_2 jm| = I_{|j_1-j_2|} \oplus I_{|j_1-j_2|+1} \oplus \cdots \oplus I_{j_1+j_2-1} \oplus I_{j_1+j_2}. \tag{78}$$

- Similarly, we can write the identity of the uncoupled representation (see the LHS of Eq. 75) as

$$\begin{aligned}&\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}|j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2| \\ &= \left(\sum_{m_1=-j_1}^{j_1}|j_1 m_1\rangle\langle j_1 m_1|\right)\otimes\left(\sum_{m_2=-j_2}^{j_2}|j_2 m_2\rangle\langle j_2 m_2|\right) \\ &= I_{j_1}\otimes I_{j_2}\end{aligned} \tag{79}$$

- Finally, by combining Eqs. 75, 78, and 79, we have the result

$$I_{j_1}\otimes I_{j_2} = I_{|j_1-j_2|}\oplus I_{|j_1-j_2|+1}\oplus\cdots\oplus I_{j_1+j_2-1}\oplus I_{j_1+j_2}\,. \tag{80}$$

- Because the direct product of two identity matrices (on the LHS) is an identity matrix, we can write the previous equation in the matrix form

$$\begin{pmatrix} 1 & & & & O \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ O & & & & 1 \end{pmatrix} = \begin{pmatrix} I_{|j_1-j_2|} & & & & O \\ & I_{|j_1-j_2|+1} & & & \\ & & \ddots & & \\ & & & I_{j_1+j_2-1} & \\ O & & & & I_{j_1+j_2} \end{pmatrix}, \tag{81}$$

where, on the LHS we have an identity matrix of dimension $(2j_1+1)(2j_2+1)$, and on the RHS, I_j denotes an identity matrix of dimension $2j+1$. Furthermore, on both the sides O denotes a null matrix block of appropriate dimensions.

- A similar result holds for the rotation matrices in the two bases

$$D^{(j_1)}(R) \otimes D^{(j_2)}(R) = D^{|j_1-j_2|}(R) \oplus D^{|j_1-j_2|+1}(R) \oplus \cdots \oplus D^{j_1+j_2-1}(R) \oplus D^{j_1+j_2}(R). \tag{82}$$

- The RHS of the previous equation implies a block-diagonal nature of the rotation matrix in the coupled basis, which can also be expressed in the matrix form similar to the case of identity matrix (Eq. 81)

$$D^{(j_1)}(R) \otimes D^{(j_2)}(R) =$$

$$\begin{pmatrix} D^{|j_1-j_2|} & & & & O \\ & D^{|j_1-j_2|+1} & & & \\ & & \ddots & & \\ & & & D^{j_1+j_2-1} & \\ O & & & & D^{j_1+j_2} \end{pmatrix} \tag{83}$$

- As a matter of fact, the main reason behind the validity of results such as Eqs. 80 and 82 is that the direct product space $\mathscr{E}_{j_1} \otimes \mathscr{E}_{j_2}$ is a direct sum of the corresponding smaller subspaces

$$\mathscr{E}_{j_1} \otimes \mathscr{E}_{j_2} = \mathscr{E}_{|j_1-j_2|} \oplus \mathscr{E}_{|j_1-j_2|+1} \oplus \cdots \oplus \mathscr{E}_{j_1+j_2-1} \oplus \mathscr{E}_{j_1+j_2}. \quad (84)$$

Examples of Block-diagonal Matrices

- We give examples of a couple of block-diagonal matrices A and B below, and how they can be written as direct sums

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \oplus \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = A_1 \oplus A_2$$

with

$$A_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}.$$

This means that the operator A is block-diagonal w.r.t. to the chosen basis in the original space $\mathscr{E}$, which can be written as the direct sum of two 2-dimensional spaces $\mathscr{E}_1$ and $\mathscr{E}_2$

$$\mathscr{E} = \mathscr{E}_1 \oplus \mathscr{E}_2.$$

Examples of Block-diagonal Matrices...

Let B be a 5×5 matrix

$$B = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 & 0 \\ 7 & 8 & 9 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \oplus \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$$

$$= B_1 \oplus B_2$$

Here clearly the 5-dimensional space $\mathscr{E}$ can be written as a direct sum of a 3-dimensional subspace $\mathscr{E}_1$ and a 2-dimensional subspace $\mathscr{E}_2$

$$\mathscr{E} = \mathscr{E}_1 \oplus \mathscr{E}_2$$

Clebsch-Gordon Series: Proof

- The exact mathematical form of the Clebsch-Gordon series, which is equivalent to Eqs. 82 and 83, is

$$D^{(j_1)}_{m_1' m_1}(R) D^{(j_2)}_{m_2' m_2}(R) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m,m'} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle \langle j_1 j_2 m_1' m_2' | j_1 j_2 j m' \rangle D^{(j)}_{m' m}(R) \tag{85}$$

Proof:

- In order to prove the result, we start with Eq. 50

$$U_R^{(\mathscr{E})} = U_R^{(\mathscr{E}_1)} \otimes U_R^{(\mathscr{E}_2)},$$

whose matrix elements in the uncoupled basis (direct-product basis) are

$$\begin{aligned}
\langle j_1 j_2 m_1' m_2' | U_R^{(\mathscr{E})} | j_1 j_2 m_1 m_2 \rangle &= \langle j_1 j_2 m_1' m_2' | U_R^{(\mathscr{E}_1)} \otimes U_R^{(\mathscr{E}_2)} | j_1 j_2 m_1 m_2 \rangle \\
&= \langle j_1 m_1' | U_R^{(\mathscr{E}_1)} | j_1 m_1 \rangle \langle j_2 m_2' | U_R^{(\mathscr{E}_2)} | j_2 m_2 \rangle \\
&= D^{(j_1)}_{m_1' m_1}(R) D^{(j_2)}_{m_2' m_2}(R)
\end{aligned}$$

Clebsch-Gordon Series: Proof...

- Using the resolution of identity in the coupled basis $\sum_{jm} |j_1 j_2 jm\rangle\langle j_1 j_2 jm| = I$ on the LHS of the previous equation two times, and interchanging LHS and RHS, we obtain

$$D^{(j_1)}_{m_1' m_1}(R) D^{(j_2)}_{m_2' m_2}(R) = \sum_{j,j',m,m'} \langle j_1 j_2 m_1' m_2' | j_1 j_2 j' m' \rangle \langle j_1 j_2 j' m' | U_R^{(\mathscr{E})} | j_1 j_2 jm \rangle \langle j_1 j_2 jm | j_1 j_2 m_1 m_2 \rangle$$

- On using the reality of CGCs $\langle j_1 j_2 jm | j_1 j_2 m_1 m_2 \rangle = \langle j_1 j_2 m_1 m_2 | j_1 j_2 jm \rangle$, and

$$\langle j_1 j_2 j' m' | U_R^{(\mathscr{E})} | j_1 j_2 jm \rangle = \delta_{j'j} D^{(j)}_{m'm}(R),$$

we obtain the desired result

$$D^{(j_1)}_{m_1' m_1}(R) D^{(j_2)}_{m_2' m_2}(R) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m,m'} \langle j_1 j_2 m_1 m_2 | j_1 j_2 jm \rangle \langle j_1 j_2 m_1' m_2' | j_1 j_2 jm' \rangle D^{(j)}_{m'm}(R)$$

Tensor Operators

- We will first discuss vector operators, and then generalize the discussion to define the tensor operators.
- Let us assume that there is an operator $\vec{A}$, called a vector operator, because its expectation value rotates as per rules of transformation of a vector, under an active rotation R.
- If the system is in state $|\psi\rangle$ we know under the rotation it will transform into $|\psi'\rangle$ as

$$|\psi'\rangle = U_R|\psi\rangle$$

- If $\hat{e}$ is an arbitrary unit vector, then clearly $\vec{A}.\hat{e}$ is a scalar and hence its value will be invariant under the rotation

$$\langle\psi|\vec{A}.\hat{e}|\psi\rangle = \langle\psi'|\vec{A}.\hat{e}'|\psi'\rangle$$

$$\langle\psi'|U_R\vec{A}.\hat{e}U_R^\dagger|\psi'\rangle = \langle\psi'|\vec{A}.\hat{e}'|\psi'\rangle$$

$$U_R\vec{A}.\hat{e}U_R^\dagger = \vec{A}.\hat{e}' \tag{86}$$

An example of an operator of the form $\vec{A}.\hat{e}$

- As an example, let us consider

$$\vec{A} = \vec{\sigma} = \sigma_x \hat{i} + \sigma_y \hat{j} + \sigma_z \hat{k}$$

$$\hat{e} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k},$$

then clearly

$$\vec{A}.\hat{e} = \frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z)$$

$$\vec{A}.\hat{e} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$$

Tensor Operator(contd.)

- $\hat{e}$ transforms into $\hat{e'}$ after the rotation. But, in a Cartesian basis

$$\hat{e}'_i = \sum_j R_{ij} \hat{e}_j \tag{87}$$

- Upon substituting this above, we obtain

$$\sum_k U_R A_k U_R^\dagger \hat{e}_k = \sum_i A_i \hat{e}'_i$$

$$\sum_k U_R A_k U_R^\dagger \hat{e}_k = \sum_i \sum_j A_i R_{ij} \hat{e}_j$$

- Comparing coefficients of $\hat{e}_j$ on both the sides, we have

$$\boxed{U_R A_j U_R^\dagger = \sum_i A_i R_{ij}} \tag{88}$$

- Let us consider an infinitesimal rotation about an axis $\hat{n}$, by an angle. Then, to the first order

$$U_R = e^{\frac{-i(\vec{J}.\hat{n})\varepsilon}{\hbar}} \approx I - \frac{i(\vec{J}.\hat{n})\varepsilon}{\hbar} \tag{89}$$

Tensor Operator(contd.)

- One can show that under such a rotation, to the first order in ε

$$\hat{e'} \approx \hat{e} + \varepsilon \hat{n} \times \hat{e} + O(\varepsilon^2) \tag{90}$$

- Substituting Eq.(90) and Eq.(89) in 86, we obtain

$$\begin{aligned}\left(I - \frac{i(\vec{J}.\hat{n})\varepsilon}{\hbar}\right)\vec{A}.\hat{e}\left(I + \frac{i(\vec{J}.\hat{n})\varepsilon}{\hbar}\right) &= \vec{A}.\hat{e} + \varepsilon\vec{A}.(\hat{n}\times\hat{e}) \\ &= \vec{A}.\hat{e} + \varepsilon(\vec{A}\times\hat{n}).\hat{e} \\ &= (\vec{A} - \varepsilon\hat{n}\times\vec{A}).\hat{e}\end{aligned}$$

- Neglecting terms O(ε^2), and comparing other terms on both sides, we obtain

$$\frac{-i}{\hbar}\varepsilon\vec{J}.\hat{n}\vec{A} + \frac{i}{\hbar}\varepsilon\vec{A}\vec{J}.\hat{n} = -\varepsilon\hat{n}\times\vec{A}$$

$$\boxed{\left[\vec{A}, \vec{J}.\hat{n}\right] = i\hbar\hat{n}\times\vec{A}} \tag{91}$$

- Using Einstein convention, we have

$$\vec{J}.\hat{n} = \sum_j J_j\hat{n}_j \equiv J_j\hat{n}_j$$

- Also,

$$(\hat{n} \times A)_i = \varepsilon_{ijk} \hat{n}_j A_k$$

- On substituting these above for the i-th component of Eq.(91), we obtain

$$\boxed{[A_i, J_j] = i\hbar \varepsilon_{ijk} A_k} \tag{92}$$

- Note that this is a very profound general relation satisfied by a vector operator $\vec{A}$
- Unlike Eq. 88, this relation (Eq. 92) does not depend on the nature of rotation (rotation axis, or the angle of rotation) in any way.
- It just involves commutation relations of various Cartesian components of $\vec{A}$ with Cartesian components of the angular momentum operator

Tensor Operators: Definition

- Next we define a tensor operator as a generalization of a vector operator.
- We saw that a vector operator $\vec{A}$ transforms according to Eq.(88) under a rotation.
- We define a spherical tensor operator T_k^q $(q = -k, -k+1, \ldots, k-1, k)$ of rank k as the operator which transforms according to the rule

$$\boxed{U_R T_k^q U_R^\dagger = \sum_{q'=-k}^{k} T_k^{q'} D_{q'q}^{(k)}(R)} \tag{93}$$

Tensor Operator(contd.)

- According to this definition, an object of rank $k=1$, is a spherical vector. Let us see how $Y_l^m(\theta,\phi)$ transform for $l=1$. We saw earlier

$$Y_l^m(\theta',\phi') = \sum_{m'=-l}^{l} Y_l^{m'}(\theta,\phi) D^{(l)}_{m',m}, \tag{94}$$

- where $Y_l^m(\theta',\phi')$ is the same function with respect to the rotated coordinate system. Noting that

$$U_R Y_l^m(\theta,\phi) U_R^\dagger = Y_l^m(\theta',\phi')$$

- we find that Eq.(93) and Eq.(94) have the same form. Thus Y_m^l's are tensor operators of rank l. Coming back to the case of $l=1$, we have

$$\begin{aligned} Y_1^{\pm 1}(\theta,\phi) &= \mp\sqrt{\frac{3}{8\pi}} e^{\pm i\phi} \sin\theta \\ &= \mp\sqrt{\frac{3}{8\pi}} \left(\sin\theta\cos\phi \pm i \sin\theta\sin\phi\right) \end{aligned}$$

Tensor Operator(contd.)

- Which can be expressed in terms of Cartesian coordinates

$$Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\frac{(x \pm iy)}{r}$$
$$Y_1^{0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r} \tag{95}$$

- So for this case Eq.(94) yields

$$\left(-\frac{x'+iy'}{\sqrt{2}}, z', \frac{x'-iy'}{\sqrt{2}}\right) = \left(-\frac{x+iy}{\sqrt{2}}, z, \frac{x-iy}{\sqrt{2}}\right) D^{(1)}_{(R)} \tag{96}$$

- Using this, we can define the components of a spherical tensor, when the Cartesian components of a vector operator are given

$$T_1^{\pm 1} = \mp\frac{A_x \pm iA_y}{\sqrt{2}}$$
$$T_1^{0} = A_z \tag{97}$$

- For an infinitesimal rotation of angle ε, about an axis along the direction $\hat{n}$, we have from Eq.(93)

$$\left(I - \frac{i}{\hbar}\vec{J}.\hat{n}\varepsilon\right) T_k^q \left(I + \frac{i}{\hbar}\vec{J}.\hat{n}\varepsilon\right) = \sum_{q'=-k}^{k} T_k^{q'} \langle kq'|I - \frac{i}{\hbar}\vec{J}.\hat{n}\varepsilon|kq\rangle$$

- Above we used the fact that $D^{(k)}_{q',q}(R) = \langle kq'|U_R|kq\rangle$, and for an infinitesimal rotation $U_R \approx I - \frac{i}{\hbar}\vec{J}.\hat{n}\varepsilon$

$$\Rightarrow T_k^q + \frac{i}{\hbar}[T_k^q, \vec{J}.\hat{n}]\varepsilon + O(\varepsilon^2) = T_k^q - \frac{i}{\hbar}\varepsilon \sum_{q'=-k} T_k^{q'} \langle kq'|\vec{J}.\hat{n}|kq\rangle$$

$$\Rightarrow \boxed{[\hat{n}.\vec{J}, T_k^q] = \sum_{q'=-k}^{k} T_k^{q'} \langle kq'|\hat{n}.\vec{J}|kq\rangle} \qquad (98)$$

Commutation Relations(contd.)

- Taking $\hat{n} = \hat{k}$, we obtain above

$$\boxed{[J_z, T_k^q] = q\hbar T_k^q} \tag{99}$$

- and $\hat{n} = \hat{n}_\pm = \hat{i} \pm i\hat{j}$, so that

$$\vec{J}.\hat{n} = J_\pm$$

- and using the fact that

$$J_\pm|kq\rangle = \hbar\sqrt{(k \mp q)(k \pm q + 1)}|kq \pm 1\rangle$$

- we obtain

$$\boxed{[J_\pm, T_k^q] = \hbar\sqrt{(k \mp q)(k \pm q + 1)}\, T_k^{q\pm 1}} \tag{100}$$

Eq.(99) and Eq.(100) are fundamental commutation relations of tensor operators.

Wigner-Eckart Theorem

- Let us compute the matrix elements of both sides of Eq. 93, with respect to angular momentum eigenstates $|\alpha jm\rangle$ and $|\alpha' j' m'\rangle$, where α, α' quantum numbers other than angular momentum, which are needed to specify these states completely

$$\langle \alpha' j' m' | U_R T_k^q U_R^\dagger | \alpha jm\rangle = \sum_{q'=-k}^{k} \langle \alpha' j' m' | T_k^{q'} | \alpha jm\rangle D_{q,q'}^{(k)}$$

- Using resolution of Identity two times on the L.H.S., we have

$$\sum_{\mu,\mu'} \langle \alpha' j' m' | U_R | \alpha' j' \mu'\rangle \langle \alpha' j' \mu' | T_k^q | \alpha j \mu\rangle \langle \alpha j \mu | U_R^\dagger | \alpha jm\rangle$$

$$= \sum_{q'=-k}^{k} \langle \alpha' j' m' | T_k^{q'} | \alpha jm\rangle D_{q,q'}^{(k)}$$

Wigner-Eckart Theorem(contd.)

- but

$$\langle \alpha' j' m' | U_R | \alpha' j' \mu' \rangle = D^{(j')}_{m',\mu'}(R)$$

$$\langle \alpha j \mu | U_R^\dagger | \alpha j m \rangle = D^{(j)*}_{m,\mu}(R)$$

- we obtain

$$\begin{aligned} &\sum_{\mu,\mu'} D^{(j')}_{m',\mu'}(R) \langle \alpha' j' \mu' | T_k^q | \alpha j \mu \rangle D^{(j)*}_{m,\mu}(R) \\ &= \sum_{q'=-k}^{k} \langle \alpha' j' m' | T_k^{q'} | \alpha j m \rangle D^{(k)}_{q,q'} \end{aligned} \tag{101}$$

- we can recast C-G series of Eq.(85) as (proof is given in problem 1 of tutorial sheet # 2)

$$\begin{aligned} &\sum_{\mu,\mu'} D^{(j')}_{m',\mu'}(R) \langle jkmq | jkj'\mu' \rangle D^{(j)*}_{m,\mu}(R) \\ &= \sum_{q'=-k}^{k} \langle jkmq' | jkj'm' \rangle D^{(k)}_{q,q'} \end{aligned} \tag{102}$$

Wigner-Eckart Theorem(contd.)

- Eq.(101) can be seen as a linear homogeneous equation for $\langle\alpha' j' m'|T_k^{q'}|\alpha j m\rangle$ and Eq.(102) has the same coefficient except that it has unknowns $\langle jkmq'|jkj'm'\rangle$.
- Thus, the solutions of two equations, must be proportional to each other. Thus

$$\langle\alpha' j' m'|T_k^{q}|\alpha j m\rangle = \langle jkmq|jkj'm'\rangle \langle\alpha' j'||T_k||\alpha j\rangle, \tag{103}$$

  we also changed $q' \to q$.
- where the proportionality constant $\langle\alpha' j'||T_k||\alpha j\rangle$ is called the **reduced Matrix element**.
- They depend only on α, α', j, j', and not on m, m', and q because that dependence is contained in the C-G-C $\langle jkmq|jkj'm'\rangle$ Eq.(103) is called the **Wigner-Eckart theorem.**

Wigner-Eckart Theorem(contd.)

- The importance of Wigner-Eckart theorem lies in the fact that the required matrix element is written as a product of C-G-C which contains the symmetry related information, and reduced matrix element which contains information about other properties of the system.

Selection Rules

- From the CGC involved in the Wigner-Eckart theorem Eq.(103), we get two important selection rules which determine whether a given matrix element is zero, based just on the symmetry. We know that the CGC $\langle jkmq|jkj'm'\rangle$ is non zero only if
  1. m selection rule is valid, i.e.,

$$m' = m + q$$

$$\Rightarrow \boxed{q = m' - m} \tag{104}$$

  2. Triangular Identity is valid, i.e.,

$$|j - k| \leq j' \leq j + k$$

  But triangular identity of numbers holds for all three numbers

$$\Rightarrow \boxed{|j - j'| \leq k \leq j + j'} \tag{105}$$

Selection Rules(contd.)

Examples:

1. For a scalar operator

$$k = 0 \Rightarrow q = 0 \Rightarrow \Delta m = m' - m = 0$$

   and

$$\Delta j = j' - j = 0 \Rightarrow j = j'$$

2. For a vector operator

$$k = 1, q = 0, \pm 1$$

$$\Rightarrow \Delta m = 0, \pm 1 \quad \textit{and} \quad \Delta j = j' - j = 0, \pm 1$$

Projection Theorem

- For a vector operator $\vec{A}$, with spherical components A_1^q or A^q for short,

$$\langle \alpha' jm' | A^q | \alpha jm \rangle = \frac{\langle \alpha' jm | \vec{J}.\vec{A} | \alpha jm \rangle}{\hbar^2 j(j+1)} \times \langle jm' | J^q | jm \rangle$$

- Proof: We have

$$\begin{aligned}
\vec{J}.\vec{A} &= J_x A_x + J_y A_y + J_z A_z \\
&= \frac{1}{2}(J_x + iJ_y)(A_x - iA_y) \\
&\quad + \frac{1}{2}(J_x - iJ_y)(A_x + iA_y) + J_z A_z \\
&= -J^{+1}A^{-1} - J^{-1}A^{+1} + J^0 A^0
\end{aligned}$$

where

$$\begin{aligned}
A^{\pm 1} &= \mp \frac{1}{\sqrt{2}}(A_x \pm iA_y) \\
A^0 &= A_z
\end{aligned}$$

- $$J^{\pm 1} = \mp \frac{1}{\sqrt{2}}(J_x \pm i J_y) = \mp \frac{1}{\sqrt{2}} J_\pm$$
  $$J^0 = J_z$$

with this

$$\begin{aligned}
\langle \alpha' jm | \vec{J}.\vec{A} | \alpha jm \rangle &= \langle \alpha' jm | J^0 A^0 - J^{+1} A^{-1} - J^{-1} A^{+1} | \alpha jm \rangle \\
&= m\hbar \langle \alpha' jm | A^0 | \alpha jm \rangle \\
&\quad + \frac{\hbar}{\sqrt{2}} \sqrt{(j+m)(j-m+1)} \langle \alpha' jm-1 | A^{-1} | \alpha jm \rangle \\
&\quad - \frac{\hbar}{\sqrt{2}} \sqrt{(j-m)(j+m+1)} \langle \alpha' jm+1 | A^{+1} | \alpha jm \rangle
\end{aligned}$$

Projection Theorem(contd.)

- But from Wigner-Eckart theorem

$$\langle \alpha' jm | A^0 | \alpha jm \rangle \propto \langle \alpha' jm | A^{+1} | \alpha jm \rangle$$

$$\propto \langle \alpha' jm | A^{-1} | \alpha jm \rangle \propto \langle \alpha' j || \vec{A} || \alpha j \rangle$$

- where $\langle \alpha' j || \vec{A} | \alpha j \rangle$ is the reduced Matrix element of $\vec{A}$, independent of m and q. Thus,

$$\langle \alpha' jm | \vec{J}.\vec{A} | \alpha jm \rangle = C(j,m) \langle \alpha' j || \vec{A} || \alpha j \rangle$$

- where $C(j,m)$ is a constant which depends on j and m, and is independent of α, α', and $\vec{A}$.
- Furthermore, $C(j,m)$ must be independent of m as well because $\vec{J}.\vec{A}$ is a scalar operator, thus

$$\langle \alpha' jm | \vec{J}.\vec{A} | \alpha jm \rangle = C(j) \langle \alpha' j || \vec{A} || \alpha j \rangle$$

Projection Theorem(contd.)

- This will be valid also for $\vec{A} = \vec{J}$, with $\alpha' = \alpha$

$$\langle \alpha jm | J^2 | \alpha jm \rangle = C(j) \langle \alpha j || \vec{J} || \alpha j \rangle$$

but $\langle \alpha jm | J^2 | \alpha jm \rangle = j(j+1)\hbar^2$

$$\Rightarrow C(j) = \frac{j(j+1)\hbar^2}{\langle \alpha j || \vec{J} || \alpha j \rangle}$$

$$\Rightarrow \langle \alpha' jm | \vec{J}.\vec{A} | \alpha jm \rangle = \frac{j(j+1)\hbar^2 \langle \alpha' j || \vec{A} || \alpha j \rangle}{\langle \alpha j || \vec{J} || \alpha j \rangle} \tag{106}$$

- using Wigner-Eckart theorem, we have

$$\Rightarrow \langle \alpha' jm' | A^q | \alpha jm \rangle = \langle j1mq | j1jm' \rangle \langle \alpha' j || \vec{A} || \alpha j \rangle$$

and

$$\Rightarrow \langle \alpha jm' | J^q | \alpha jm \rangle = \langle j1mq | j1jm' \rangle \langle \alpha' j || \vec{J} || \alpha j \rangle$$

Projection Theorem(contd.)

- Taking the ratio of these two equations, we have

$$\frac{\langle \alpha' j || \vec{A} | \alpha j \rangle}{\langle \alpha' j || \vec{J} | \alpha j \rangle} = \frac{\langle \alpha' j m' | A^q | \alpha j m \rangle}{\langle \alpha j m' | J^q | \alpha j m \rangle} \tag{107}$$

- on substituting Eq.(107) in Eq.(106) we obtained the desired result

$$\langle \alpha' j m' | A^q | \alpha j m \rangle = \frac{\langle \alpha' j m | \vec{J}.\vec{A} | \alpha j m \rangle}{j(j+1)\hbar^2} \times \langle j m' | J^q | j m \rangle \tag{108}$$

- The importance of projection theorem is that it shows that the expectation value of any vector operator is proportional to the expectation value of the angular momentum operator.
- This implies that any vector associated with a spherically symmetric quantum mechanical system will be either parallel or anti parallel to its angular momentum.