$\Rightarrow$

$$c'_k = \sum_i \sum_j A_{kj} B_{ji} c_i$$

Define a matrix

$$\boxed{D_{ki} = \sum_j A_{kj} B_{ji}}$$

$\Rightarrow$

$$c'_k = \sum_k D_{ki} c_i$$

$\Rightarrow$

$$\tilde{C}' = D \tilde{C}$$

$D = AB$ is the matrix representation of the composite operator TU. The matrices A and B are multiplied as per the rules of matrix multiplication given above.

(iii) <u>Representation of the identity Operator</u> :

We know that identity operator I satisfies

$$I\alpha = \alpha \qquad \forall\, \alpha \in V$$

$\Rightarrow$

$$I\alpha_i = \sum_j A_{ji}\alpha_j = \alpha_i \qquad \forall\, i \leq n$$

$\Rightarrow$ $\Rightarrow$ $\boxed{A_{ji} = \delta_{ij}}$

$\Rightarrow$ representation of I is the $n \times n$ identity matrix.

Change of Basis:

We saw that the representation of vectors and operators in a vector space depends upon the basis chosen. The question we pose next is that if change the basis ~~of~~ set of our vector space, how will the representation of various operators and vectors change.

Let $B^{\circ} = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and $B' = \{\alpha_1', \alpha_2', \ldots \alpha_n'\}$ be two possible basis sets in the space V. Since they are vectors themselves, one can write the vectors of one set in terms of the other, i.e.,

$$\alpha_i' = \sum_j P_{ji}\, \alpha_j \quad \text{— (17a)}$$

and

$$\alpha_i = \sum_j P'_{ji}\, \alpha_j' \quad \text{— (17b)}$$

using (17a) in (17b)

$$\cancel{\alpha_i = \sum_j P_{ji} \sum_k P_{kj}}$$

$$\alpha_i = \sum_j P'_{ji} \sum_k P_{kj}\, \alpha_k$$

$$\alpha_i = \sum_k \Big(\sum_j P_{kj} P'_{ji} \Big) \alpha_k$$

comparing both sides

$$\Rightarrow \quad \sum_j P_{kj} P'_{ji} = \delta_{ki}$$

$$\Rightarrow \quad PP' = I$$

similarly substituting (17b) in (17a) we obtain

$$P'P = I$$

$\Rightarrow$ P' & P are inverses of each other

$$\boxed{P' = P^{-1}} \qquad \text{—— (18)}$$

$\Rightarrow$ a change of basis transformation must be invertible.

(i) Change of basis for a vector :

Let $\alpha \in V$ and w.r.t. $B = \{\alpha_1, \alpha_2, \alpha_n$ it can be represented as

$$\alpha = \sum_i x_i \alpha_i \qquad \text{—— (19)}$$

we want to obtain α w.r.t. $B' = \{\alpha_1', \alpha_2', .. \alpha_n'\}$, i.e.

$$\alpha = \sum_i x_i \alpha_i' \qquad \text{—— (20)}$$

using 17b in 19, we have

$$\alpha = \sum_i x_i \sum_j P^{-1}_{ji} \alpha_j'$$

$$\alpha = \sum_j \left(\sum_i P^{-1}_{ji} x_i \right) \alpha_j' \qquad \text{— (21)}$$

comparing 20 & 21, we have

$$x_i' = \sum_j P^{-1}_{ij} x_j$$

or

$$\boxed{X' = P^{-1} X} \qquad \text{— (22)}$$

where

$$X' = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} P^{-1}_{11} & \cdots & P^{-1}_{1n} \\ \vdots & & \\ P_{n1} & \cdots & P^{-1}_{nn} \end{pmatrix}$$

Eq. (22) is the desired relationship.

(ii) Change of basis for an operator :

Let $T : V \to V$ and $B = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ and $B' = \{\alpha_1', \alpha_2', \dots \alpha_n'\}$ be two bases. With respect to B and B' the representations of T are

$$T\alpha_i = \sum_j A_{ji}\alpha_j \qquad \text{— (23)}$$

$$T\alpha_i' = \sum_j A_{ji}'\alpha_j' \qquad \text{— (24)}$$

Our aim is to obtain a relationship between A and A' representations.

$$T\alpha_i' = T\left(\sum_j P_{ji}\alpha_j\right)$$

$$= \sum_j P_{ji}\, T\alpha_j$$

$$= \sum_j P_{ji} \sum_k A_{kj}\,\alpha_k$$

but $\alpha_k = \sum_l P^{-1}_{lk}\alpha_l'$

$$\Rightarrow T\alpha_i' = \sum_j P_{ji} \sum_k A_{kj} \sum_l P^{-1}_{lk}\alpha_l'$$

$$T\alpha_i' = \sum_{jkl} P^{-1}_{lk} A_{kj} P_{ji}\,\alpha_l' \qquad \text{— (25)}$$

$$T\alpha_i' = \sum_l \left(\sum_{jk} P^{-1}_{lk} A_{kj} P_{ji}\right)\alpha_l'$$

--

Comparing 24 & 25

$$A'_{li} = \sum_{jk} P^{-1}_{lk} A_{kj} P_{ji}$$

or $$\boxed{A' = P^{-1} A P} \quad \text{—— } 26$$

This equation establishes the connection between the two representations of the operator T.

~~A~~ A transformation of the form 26 is called a similarity transformation and any two ~~po~~ operators connected by such a transformation are called similar.

Operator Representation in an Orthogonal Basis:

For $T: V \to V$ and with a basis

$B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$

$$T\alpha_i = \sum_k A_{ki}\alpha_j$$

if the basis is orthogonal

$$(\alpha_j | T\alpha_i) = (\alpha_j | \sum_k A_{ki}\alpha_k) = \sum_k A_{ki} (\alpha_j | \alpha_k)$$

for an orthonormal basis

$$(\alpha_j | \alpha_k) = \delta_{jk}$$

$$\Rightarrow \boxed{(\alpha_j | T\alpha_i) = A_{ji}} \quad \text{--- (27)}$$

~~Thus using~~ Thus on an I.P. space, for an orthonormal basis, one can obtain the ~~[illegible]~~ representation of the operator T using the formula given above.

Hermitian Adjoint (Conjugate) of an Operator:

Next we introduce the concept of the Hermitian adjoint/conjugate T^+ of a linear operator T.

<u>Theorem</u>: For every linear operator T on a finite dimensional vector space V, there exists a unique linear operator T^+ on V such that

$$(\beta|T\alpha) = (T^+\beta|\alpha) \quad \forall\, \alpha, \beta \in V. \tag{28}$$

Proof: Using

$$T\alpha_i = \sum_j A_{ji}\alpha_j$$

$$\alpha = \sum_i x_i\alpha_i$$

$$\beta = \sum_i y_i\alpha_i$$

one can show that

$(\beta|T\alpha)$ can be rewritten as $(T^+\beta|\alpha)$

when one defines

$$T^+\alpha_i = \sum_j B_{ji}\alpha_j$$

and

$$B_{ji} = A^*_{ij}$$

so ~~if~~ the matrix representations of the two operators are also Hermitian conjugates of each other. One can also show that

$$T^+(c\beta + \gamma) = cT^+\beta + T^+\gamma \quad — (29)$$

completing the proof that T^+ is a linear operator.

Properties of Hermitian Adjoint :

One can similarly show

$$\left.\begin{aligned} &(i) \quad (T+U)^+ = T^+ + U^+ \\ &(ii) \quad (cT)^+ = c^* T^+ \\ &(iii) \quad (TU)^+ = U^+ T^+ \\ &(iv) \quad (T^+)^+ = T \end{aligned}\right\} \quad — (30)$$

Hermitian Operators:

An operator $T: V \to V$ is said to be Hermitian if it is equal to its Hermitian adjoint, i.e.

$$\boxed{T^{\dagger} = T}$$

which, for the representations implies

$$\boxed{A_{ji}^{*} = A_{ij}}$$

such matrices are called Hermitian matrices.

Unitary Operators:

Let $U: V \to V$ be a linear operator on the I.P. space V. U is called a linear operator if it preserves the I.P. on V, i.e.,

$$(U\alpha | U\beta) = (\alpha | \beta) \quad \forall \alpha, \beta \in V$$

Note that it follows from above that

$$|U\alpha| = |\alpha|$$

i.e. U conserves the norm of every vector in V.

One can easily show that for all unitary operators U

$$U^{+}U = UU^{+} = I$$

$$\Rightarrow \quad U^{+} = U^{-1}$$