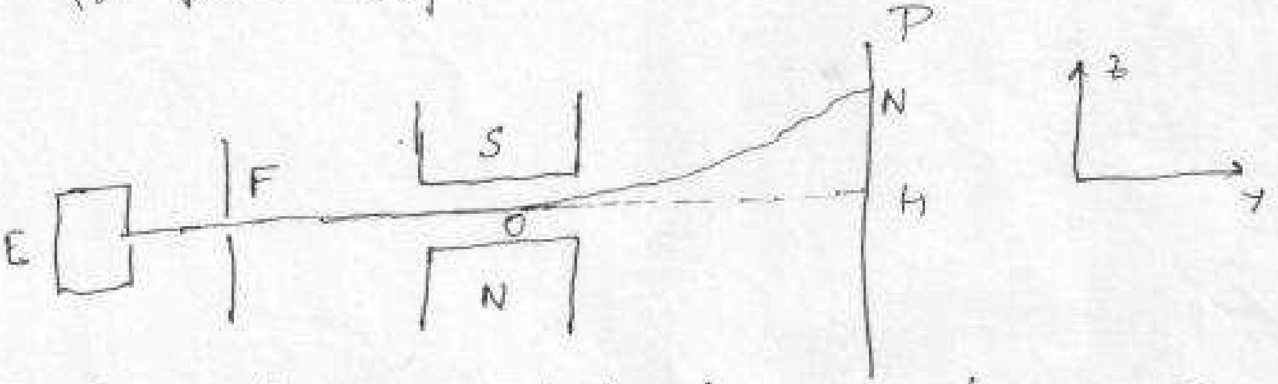
# Chapter 5
## Spin-$\frac{1}{2}$ and two-level Systems

In this chapter we will illustrate the postulates of quantum mechanics by applying them ~~to~~ to measurements performed on spin-$\frac{1}{2}$ particles and other two-level systems.

## Stern-Gerlach Experiment & the Quantization of the Angular Momentum:

In this experiment paramagnetic atoms such as ~~the~~ silver atoms (which have net angular momentum of ½) are exposed to a highly inhomogeneous field ~~and~~ leading to their deflection as shown below.




Silver atoms contained in a furnace E are ~~heated~~ heated to high temperatures and are let out ~~thr~~ through a small hole. Once these atoms pass through a collimating slit, they travel in the shape of a beam with their velocities parallel to the y-direction denoted in the figure. Next they pass through a highly inhomogeneous magnetic field in the z-direction produced by an electromagnet. As a result they experience a force in the z-direction and get deflected in that direction and hit the plate P where they are detected.

Classical Predictions of the Deflection:

Since atoms are neutral they do not experience Lorentz force in the B-field. However, they do possess a magnetic moment $\vec{\mu}$ because of their paramagnetic nature and thus experience the potential

$$U = -\vec{\mu}\cdot\vec{B} \quad (1)$$

and because $\vec{B}$ is inhomogeneous, $U$ will be position dependent and will thus lead to a force causing the deflection

It can be shown that the magnetic moment of an atom is proportional to its instrinsic angular momentum $\vec{L}$

$$\vec{\mu} = \gamma \vec{L} \quad \text{——} \quad (2)$$

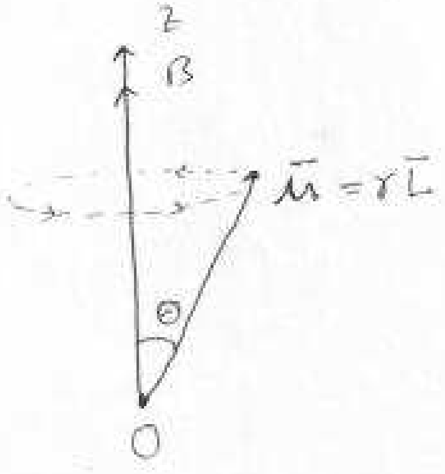
where $\gamma$ is called the ~~gyromom~~ gyromagnetic ratio of the material. The force experienced by the atoms will be

$$\vec{F} = -\vec{\nabla} U = \vec{\nabla}(\vec{\mu}\cdot\vec{B}) \quad \text{——} \quad (3)$$

and the torque experienced will be

$$\vec{\tau} = \vec{\mu} \times \vec{B} \quad \text{——} \quad (4)$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{\tau} = \vec{\mu} \times \vec{B} = \gamma(\vec{L} \times \vec{B}) \quad \text{——} \quad (5)$$




This torque, which is perpendicular to both $\vec{B}$ and $\vec{L}$, causes $\vec{L}$ process about

$\vec{B}$ with the angular velocity $\gamma|B|$, keeping angle $\theta$ constant. Thus component of $\vec{L}$ perpendicular to $\vec{B}$ rotates while the one along $\vec{B}$ remains constant.

The force $\vec{F}$ on the atom to a very good approximation is

$$\vec{F} = \vec{\nabla}(\mu_z B_z) = \mu_z \vec{\nabla} B_z \quad \text{—} (6)$$

~~because $B_x$ and $B_y$ are negligible~~

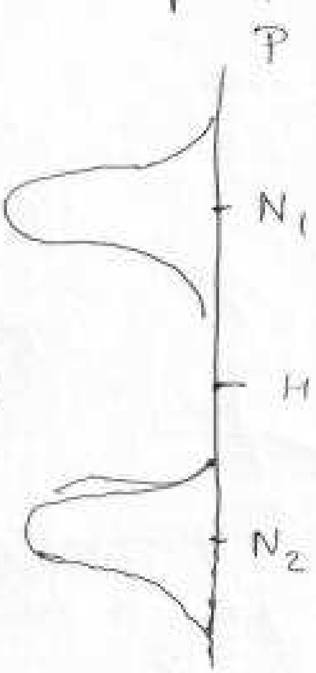
because contributions of $\mu_x$ & $\mu_y$ average out to zero because of their rapid rotation. Finally

$$\vec{F} = F\hat{k} = \mu_z \frac{\partial B_z}{\partial z}\hat{k} \quad \text{—} (7)$$

because the variation of $B_z$ is negligible in $x$ and $y$ directions. Therefore the atoms of the beam will experience deflection in the $z$ direction proportional to the value of $\mu_z$. Because $\mu_z$ is expected to vary continuously from $-|\mu|$ to $+|\mu|$, we expect a pattern of atoms striking the plate to be symmetrical with respect to H.

## Results & Inferences:

The results of this experiment performed by Stern & Gerlach (1922) are in complete contradiction with the classical predictions.




Instead of obtaining a single spot at H, we obtain two spots located at points $N_1$ and $N_2$, placed symmetrical with respect to H.

The only way this result can be explained is if we assume that $\mu_z$ takes only two possible values one positive and the other one negative leading to two possible deflections of the beam leading to spots at $N_1$ and $N_2$.

The exposure of the atoms to the B-field directed along the z direction amounts to measuring $\mu_z$ component of the atoms which will yield one of the two possible values as the result of the measurement. Thus Stern-Gerlach experiment ~~results~~ can be explained only by the quantum nature of the angular momenta of objects such as atoms. Later on we will learn that atoms such as Ag atoms are spin-$\frac{1}{2}$ objects ~~whose which can only take~~ whose angular momentum components take values $\pm\hbar/2$.

## Quantum Mechanical Theory:

With $L_z$ of these atoms we associate the observable $S_z$, and with its two possible eigenvalues $\pm\frac{\hbar}{2}$, we associate eigenvectors $|\pm\rangle$ such that

$$\left.\begin{aligned} S_z|+\rangle &= \frac{\hbar}{2}|+\rangle \\ S_z|-\rangle &= -\frac{\hbar}{2}|-\rangle \end{aligned}\right\} \quad (8)$$

so that

$$\left.\begin{aligned} \langle +|+\rangle &= \langle -|-\rangle = 1 \\ \langle +|-\rangle &= \langle -|+\rangle = 0 \end{aligned}\right\} \quad (9)$$

and

$$|-\rangle\langle -| + |+\rangle\langle +| = \mathbb{1} \quad (10)$$

~~Obviously~~ Thus $S_z$ is an operator in a two-dimensional state space and

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (11)$$

We will have to associate different observables with $L_x$ and $L_y$ components of the angular momenta. We shall learn later that $L_x$, $L_y$, & $L_z$ do not commute with each other and using the proper commutation relations one can obtain the representation of the observables corresponding to $L_x$ and $L_y$ as

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{——— (12)}$$

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{——— (13)}$$

One can have angular momentum component in an arbitrary direction ~~depend on~~ $\hat{u}$ defined by polar angles $(\theta, \phi)$

$$\hat{u} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k} \quad \text{— (14)}$$

with the corresponding angular momentum component

$$L_u = \bar{L}\cdot\hat{u} = L_x\sin\theta\cos\phi + \sin\theta\sin\phi\, L_y + \cos\theta\, L_z$$

we will have the corresponding observable

$$S_u = S_x\sin\theta\cos\phi + S_y\sin\theta\sin\phi + S_z\cos\theta \quad \text{—— (15)}$$

$$S_u = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \quad \text{——— (16)}$$

It is easy to show that the ~~eigenvectors~~ eigenvalues ~~$S_x, S_y$~~ of $S_x$, $S_y$, and $S_n$ are also $\pm\frac{\hbar}{2}$.

The eigenvectors of $S_n$ can be obtained to be

$$|+\rangle_n = \cos\frac{\theta}{2}\, e^{-i\phi/2}|+\rangle + \sin\frac{\theta}{2}\, e^{i\phi/2}|-\rangle \quad \text{— (17a)}$$

$$|-\rangle_n = -\sin\theta/2\, e^{-i\phi/2}|+\rangle + \cos\theta/2\, e^{i\phi/2}|-\rangle \quad \text{— (17b)}$$

## Preparation of states and ~~sub~~ measurements in the Stern-Gerlach Experiment:

We saw that exposing ~~to~~ a beam of silver atoms to a inhomogeneous in the z direction amounts to a measurement of the observable $S_z$ with either of $\pm\hbar/2$ results possible, leading to the splitting of the beam into two beams corresponding to those two values. If we now choose one of those beam we know that all the atoms of that beam are in one state $|+\rangle$ or $|-\rangle$. Thus, for subsequent measurements, our initial state $|\psi\rangle$ has been prepared and is known to be $|+\rangle$ or $|-\rangle$ depending upon the chosen beam.

Next we can arrange another magnetic field oriented in a different direction (in general $S_u$) through which the chosen beam passes. This way we will obtain either of $\pm\hbar/2$ as ~~doing~~ the result of the measurement as long as direction $\hat{u}$ is not collinear with the direction z. The probability of the either result will be as predicted by postulate 4 of quantum mechanics.

As one can imagine, one can perform a series of measurements on subsequent beam of atoms to verify the postulates of quantum mechanics.

We will illustrate this discussion by analysing ~~performing~~ a situation where the first measurement is of $S_z$ leading $\pm\frac{\hbar}{2}$ values and the reduced wave packets $|\pm\rangle$. Then the subsequent measurement is performed on $|+\rangle$ beam atoms using a field oriented along the x-axis. Thus the result of the measurement will be an eigenvalue of $S_x$

which will be $\pm\frac{\hbar}{2}$ oriented along the $x$-axis. ~~Note that class~~ We will compute the corresponding probabilities using the postulates of QM. Note, however, from a classical point of view a beam with $\vec{\mu} = +\frac{\hbar}{2}\hat{e}_z$, will not yield any positive result for a $\vec{\mu}$ along $x$-axis because the two directions are mutually perpendicular.

Let us compute the probabilities. Here

$$|\psi\rangle = |+\rangle \quad \text{--- (18)}$$

The observable measured is

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{--- (19)}$$

whose eigenvalues are

$$\lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$$

$$\Rightarrow \lambda_{1,2} = \pm\frac{\hbar}{2} \quad \text{--- (20)}$$

with the eigenvectors

$$\left.\begin{aligned} |+\rangle_x &= \frac{1}{\sqrt{2}}\{|+\rangle + |-\rangle\} \\ \text{and} \quad |-\rangle_x &= \frac{1}{\sqrt{2}}\{|+\rangle - |-\rangle\} \end{aligned}\right\} \quad \text{--- (21)}$$

As per postulate 4 of QM, the probability of measuring $+\frac{\hbar}{2}$ along the $x$-axis will be

$$P_{+x} = |{}_x\langle +|\psi\rangle|^2 = \left| \frac{1}{\sqrt{2}}\langle +|+\rangle + \frac{1}{\sqrt{2}}\underbrace{\langle -|+\rangle}_{0} \right|^2$$

$$P_{+x} = \frac{1}{2}$$

similarly

$$P_{-x} = |{}_x\langle -|\psi\rangle|^2 = \frac{1}{2}$$

Thus the quantum mechanical probability of measuring $\pm\frac{\hbar}{2}$ spin in the $x$-direction, if the initial spin is polarised along the $z$-direction, is $\frac{1}{2}$ each. This is totally contrary to the classical expectations.

~~Spin-½~~

Quantum Mechanical Theory of Spin-$\frac{1}{2}$ Particles in Magnetic Fields:

(i) Case of static uniform Magnetic Field:

We know that the classical energy of a magnetic moment in a magnetic field is

$$W = -\bar{\mu}\cdot\bar{B}_0$$

if $\vec{B}_0 = B_0 \hat{e}_z$

and $\vec{\mu} = \gamma \vec{L}$

$$W = -\gamma L_z B_0$$

let $\omega_0 = -\gamma L_z$

$$W = \omega_0 L_z$$

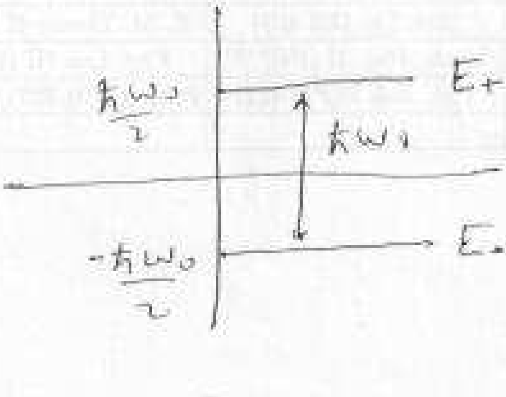
so as per the ~~qua~~ quantization rules the QM Hamiltonian will be

$$H = \omega_0 S_z = \begin{pmatrix} \frac{\hbar\omega_0}{2} & 0 \\ 0 & -\frac{\hbar\omega_0}{2} \end{pmatrix} \quad \text{— (22)}$$

which as eigenvectors $|\pm\rangle$

$$\left.\begin{aligned} H|+\rangle &= \frac{\hbar\omega_0}{2}|+\rangle \\ H|-\rangle &= -\frac{\hbar\omega_0}{2}|-\rangle \end{aligned}\right\} \quad \text{— (23)}$$

Thus the two energy levels of the system are



Let us assume that the the initial state of the system has spin directed along $(\theta, \phi)$ direction

$$|\psi(0)\rangle = \cos\frac{\theta}{2}\, e^{-i\phi/2} |+\rangle + \sin\frac{\theta}{2}\, e^{i\phi/2} |-\rangle$$

then the time evolution of the system will be

$$|\psi(t)\rangle = \cos\frac{\theta}{2}\, e^{-i\phi/2} e^{-iE_+ t/\hbar} |+\rangle + \sin\frac{\theta}{2}\, e^{i\phi/2} e^{-iE_- t/\hbar} |-\rangle$$

$$|\psi(t)\rangle = \cos\frac{\theta}{2}\, e^{-i(\phi+\omega_0 t)/2} |+\rangle + \sin\frac{\theta}{2}\, e^{i(\phi+\omega_0 t)/2} |-\rangle$$

thus effectively for this system

$$\theta(t) = \theta$$

$$\phi(t) = \phi$$

$$|\psi(t)\rangle = \cos\frac{\theta}{2}\, e^{-i(\phi+\omega_0 t)/2} |+\rangle + \sin\frac{\theta}{2}\, e^{i(\phi+\omega_0 t)/2} |-\rangle$$

thus it is as though for this spin

$$\theta(t) = \theta$$

$$\phi(t) = \phi + \omega_0 t$$

that is spin is precessing about the magnetic field direction ($z$ axis) at constant $\theta$, and with constant angular velocity $\omega_0$. This is the quantum analogue of Larmor precession.

## (ii) Theory of Magnetic Resonance

We saw that when atoms with magnetic moments are exposed to static magnetic field, ~~they begin to~~ their magnetic moments begin to precess about the magnetic field with angular velocity $\omega_0 = -\gamma |B_0|$ ($|B_0|$ is the strength of the field). The question is: if we subject this precessing magnetic moment to a magnetic ~~rotating~~ field which is rotating with an angular velocity $\omega \approx \omega_0$, will we observe a resonance phenomenon?

We will address this issue next.

Let us assume that a magnetic moment

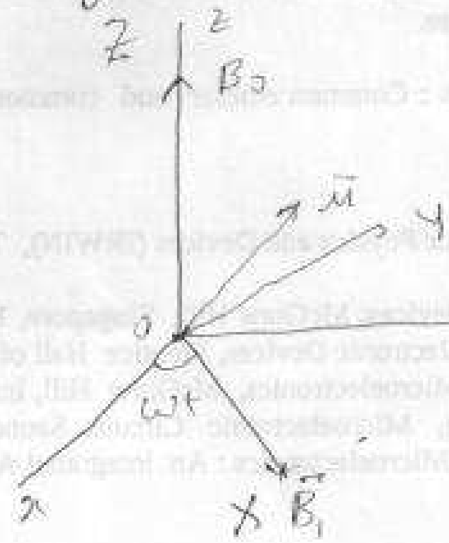
$$\vec{\mu} = \gamma \vec{L}$$

~~is exp~~

is exposed to a magnetic field $\vec{B}_0 = B_0\hat{k}$ and a rotating magnetic field which is perpendicular to $\vec{B}_0$

$$\vec{B}_1 = B_1\cos\omega t\,\hat{i} + B_1\sin\omega t\,\hat{j} \qquad (25)$$

where $\omega$ is the angular frequency of the rotation of $\vec{B}_1$.



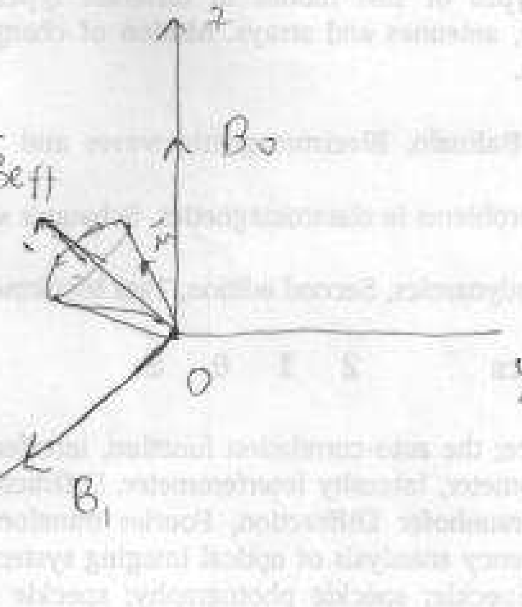

so that the equations of motion of $\vec{\mu}$ is

$$\frac{d\vec{\mu}}{dt} = \gamma\frac{d\vec{L}}{dt} = \gamma\,\vec{\mu}\times(\vec{B}_0+\vec{B}_1) \qquad (26)$$

Now if we define a frame which rotates with $\vec{B}$ and is defined by directions OX, OY, and OZ so that $\vec{B}_1$ is in the X direction, and OZ and Oz directions are coincidental. With respect to this frame $\vec{B}_1$ and $\vec{B}_0$ are both ~~fixed~~ time independent. ~~that B is fixed~~

~~in space~~

That is spatial directions of both these fields is fixed with respect to the OXYZ fram. ~~These two fields then give rise to a~~




~~net field $\bar{B}_{eff}$ in the OXZ plane.~~

Using the relation between the time derivatives in the fixed and rotating frames

$$\frac{d}{dt} = \left(\frac{d}{dt}\right)_{rot} + \bar{\omega}\times$$

we obtain

$$\left(\frac{d\bar{\mu}}{dt}\right)_{rot} = \frac{d\mu}{dt} - \bar{\omega}\times\bar{\mu}$$

$$\cancel{\left(\frac{d\mu}{dt}\right)_{rot} = \gamma\bar{\mu}\times\bar{B}_{eff} - \bar{\omega}\times\bar{\mu}}$$

$$\left(\frac{d\vec{\mu}}{dt}\right) = \gamma \vec{\mu} \times (B_0 \hat{e}_z + B_1 \hat{e}_x) + \cancel{B_0} \omega \vec{\mu} \times$$

define $\hat{e}_x = \hat{i}\cos\omega t + \hat{j}\sin\omega t$

$$\left(\frac{d\vec{\mu}}{dt}\right)_{rot} = \gamma \vec{\mu} \times (B_0 \hat{e}_z + B_1 \hat{e}_x) + \vec{\mu} \times (\omega \hat{e}_z) \quad \text{--- (27)}$$

defining $\omega_0 = -\gamma B_0$

$\omega_1 = -\gamma B_1$

$$\left(\frac{d\vec{\mu}}{dt}\right)_{rot} = \vec{\mu} \times (\Delta\omega \hat{e}_z - \omega_1 \hat{e}_x) \quad \text{--- (28)}$$

or where $\Delta\omega = \omega - \omega_0$

$$\Rightarrow \left(\frac{d\vec{\mu}}{dt}\right)_{rot} = \gamma \vec{\mu} \times \vec{B}_{eff} \quad \text{--- (29)}$$

where

$$\boxed{\vec{B}_{eff} = \frac{\Delta\omega \hat{e}_z - \omega_1 \hat{e}_x}{\gamma}} \quad \text{--- (30)}$$

Thus in the OXYZ frame $\vec{\mu}$ is precessing about an effective field $\vec{B}_{eff}$ which is in the XZ plane. There are two possibilities:

(i) When $\Delta\omega >> \omega_1$, $\vec{B}_{eff}$ is along OZ and thus $\vec{\mu}$ precesses effectively about OZ axis as before. This is the off resonance situation

(ii) $\Delta\omega = 0$, i.e., $\omega = \omega_0$ $\Rightarrow$ natural frequency of the system is same as the external frequency. Here $B_{eff} \parallel \hat{e}_x$ and thus $\vec{\mu}$ precesses about the x axis leading to a probability of spin flip.

~~This the situation of the process magnetic resonance~~

This is the phenomenon of magnetic resonance

## Quantum Theory of Magnetic Resonance:

The Hamiltonian for this system is

$$H = -\vec{\mu}\cdot(\vec{B}_0 + \vec{B}_1) = -\gamma\,(B_0 S_z + B_1 \cos\omega t\, S_x + B_1 \sin\omega t\, S_y)$$

$$H = \omega_0 S_z + \omega_1 \cos\omega t\, S_x + \omega_1 \sin\omega t\, S_y$$

$$H = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix}$$

The Schrödinger equation is

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

if $|\psi\rangle = \begin{pmatrix} a_+(t) \\ a_-(t) \end{pmatrix}$

$$i\hbar \begin{pmatrix} \frac{da_+}{dt} \\ \frac{da_-}{dt} \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix} \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$i\frac{da_+}{dt} = \frac{\omega_0}{2} a_+(t) + \frac{\omega_1}{2} e^{-i\omega t} a_-$$

$$i\frac{da_-}{dt} = \frac{\omega_1}{2} e^{+i\omega t} a_+(t) - \frac{\omega_0}{2} a_-(t)$$

We can transform to a rotating frame by defining

$$b_+(t) = a_+ e^{i\omega t/2}$$

$$b_-(t) = a_- e^{-i\omega t/2}$$

$$\Rightarrow \quad a_+(t) = b_+(t) e^{-i\omega t/2}$$

$$a_-(t) = b_-(t) e^{i\omega t/2}$$

$$\Rightarrow \quad \frac{da_+}{dt} = e^{-i\omega t/2} \frac{db_+}{dt} - \frac{i\omega}{2} b_+(t) e^{-i\omega t/2}$$

and

$$\frac{da_-}{dt} = e^{i\omega t/2} \frac{db_-}{dt} + \frac{i\omega}{2} e^{i\omega t/2} b_-$$

$$\Rightarrow \quad i e^{-i\omega t/2} \frac{db_+}{dt} + \frac{\omega}{2} e^{-i\omega t/2} b_+(t)$$

$$= \frac{\omega_0}{2} e^{-i\omega t/2} b_+(t) + \frac{\omega_1}{2} e^{-i\omega t/2} b_-(t)$$

$$\Rightarrow \quad i \frac{db_+}{dt} = -\frac{\Delta\omega}{2} b_+ + \frac{\omega_1}{2} b_-(t)$$

and similarly

$$i\frac{db_-}{dt} = \frac{\omega_1}{2} b_+ + \frac{\Delta\omega}{2} b_-(t)$$

by defining $|\tilde{\Psi}(t)\rangle = \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}$

The equations above reduce to

$$i\hbar \frac{d|\tilde{\Psi}(t)\rangle}{dt} = \tilde{H} |\tilde{\Psi}(t)\rangle$$

with $\tilde{H} = \frac{\hbar}{2} \begin{pmatrix} -\Delta\omega & \omega_1 \\ \omega_1 & \Delta\omega \end{pmatrix} \equiv$ time independent

Next we obtain the eigen values ~~and the~~ ~~eigenvectors~~ of $\tilde{H}$.

Eigenvalues of $\tilde{H}$ are readily obtained

$$E^{\pm} = \pm \frac{\hbar}{2}\sqrt{\omega_1^2 + \Delta\omega^2} = \pm \frac{\hbar}{2}\omega_R$$

when $\omega_R = \sqrt{\omega_1^2 + \Delta\omega^2} \equiv$ Rabi Frequency

We want to compute the probability that if the initial state of the system was spin up

$$|\Psi(0)\rangle = |\tilde{\Psi}(0)\rangle = |+\rangle$$

at a given time $t$, the system will be found in the spin down state $|-\rangle$.

To compute that we need to know $|\Psi(t)\rangle$ which can be computed from $|\tilde{\Psi}(t)\rangle$. We compute $|\tilde{\Psi}(t)\rangle$ using time evolution operator.

$$|\tilde{\Psi}(t)\rangle = \tilde{U}(t,0)|\tilde{\Psi}(0)\rangle = e^{-i\tilde{\mathcal{H}}t/\hbar}|\Psi(0)\rangle$$

Now

$$e^{-i\tilde{\mathcal{H}}t/\hbar} = I - i\tilde{\mathcal{H}}t/\hbar + \frac{(-i\tilde{\mathcal{H}}t)^2}{2\hbar^2} + \cdots$$

but

$$\tilde{\mathcal{H}}^2 = \frac{\hbar^2}{4}\begin{pmatrix} -\Delta\omega & \omega_1 \\ \omega_1 & \Delta\omega \end{pmatrix}\begin{pmatrix} -\Delta\omega & \omega_1 \\ \omega_1 & \Delta\omega \end{pmatrix}$$

So

$$\tilde{\mathcal{H}}^2 = \frac{\hbar^2}{4}\begin{pmatrix} \omega_1^2 + \Delta\omega^2 & 0 \\ 0 & \Delta\omega^2 + \omega_1^2 \end{pmatrix} = \frac{\hbar^2 \omega_R^2}{4} I$$

so

$$\tilde{\mathcal{H}}^{2n} = \left(\frac{\hbar\omega_R}{2}\right)^{2n} I$$

$$\tilde{\mathcal{H}}^{2n+1} = \left(\frac{\hbar\omega_R}{2}\right)^{2n} \tilde{\mathcal{H}}$$

so

$$\tilde{U}(t,0) = e^{-i\tilde{\mathcal{H}}t/\hbar} = \left(I - \frac{1}{2!}\left(\frac{\omega_R t}{2}\right)^2 + \frac{1}{4!}\left(\frac{\omega_R t}{2}\right)^4 + \cdots\right) I$$

$$- \frac{2i\mathcal{H}}{\hbar\omega_R}\left\{\frac{\omega_R t}{2} - \frac{1}{3!}\left(\frac{\omega_R t}{2}\right)^3 + \cdots\right\}$$

$$= \cos\frac{\omega_R t}{2}\, I - \frac{2i\tilde{\mathcal{H}}}{\hbar\omega_R}\sin\frac{\omega_R t}{2}$$

$$= \begin{pmatrix} \cos\omega_R t/2 & 0 \\ 0 & \cos\frac{\omega_R t}{2} \end{pmatrix} - i\begin{pmatrix} -\frac{\Delta\omega}{\omega_R}\sin\frac{\omega_R t}{2} & \frac{\omega_1}{\omega_R}\sin\frac{\omega_R t}{2} \\ \frac{\omega_1}{\omega_R}\sin\frac{\omega_R t}{2} & \frac{\Delta\omega}{\omega_R}\sin\frac{\omega_R t}{2} \end{pmatrix}$$

$$\Rightarrow \tilde{U}(t,0) = \begin{pmatrix} \cos\frac{\omega_R t}{2} + i\frac{\Delta\omega}{\omega_R}\sin\frac{\omega_R t}{2} & -i\frac{\omega_1}{\omega_R}\sin\frac{\omega_R t}{2} \\ -i\frac{\omega_1}{\omega_R}\sin\frac{\omega_R t}{2} & \cos\frac{\omega_R t}{2} - \frac{i\Delta\omega}{\omega_R}\sin\frac{\omega_R t}{2} \end{pmatrix}$$

Now

$$|\tilde{\psi}(t)\rangle = \tilde{U}(t,0)\,|+\rangle$$

$$\equiv \begin{pmatrix} \cos\frac{\omega_R t}{2} + i\frac{\Delta\omega}{\omega_R}\sin\frac{\omega_R t}{2} & -i\frac{\omega_1}{\omega_R}\sin\frac{\omega_R t}{2} \\ -i\frac{\omega_1}{\omega_R}\sin\frac{\omega_R t}{2} & \cos\frac{\omega_R t}{2} - \frac{i\Delta\omega}{\omega_R}\sin\frac{\omega_R t}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or

$$|\tilde{\psi}(t)\rangle \equiv \begin{pmatrix} \cos\frac{\omega_R t}{2} + i\frac{\Delta\omega}{\omega_R}\sin\frac{\omega_R t}{2} \\ -\frac{i\omega_1}{\omega_R}\sin\frac{\omega_R t}{2} \end{pmatrix}$$

$$\Rightarrow \quad |\psi(t)\rangle = \begin{pmatrix} \left\{\cos\frac{\omega_R t}{2} + i\frac{\Delta\omega}{\omega_R}\sin\frac{\omega_R t}{2}\right\} e^{-i\omega t/2} \\ -i\frac{\omega_1}{\omega_R}\sin\frac{\omega_R t}{2}\, e^{i\omega t/2} \end{pmatrix}$$

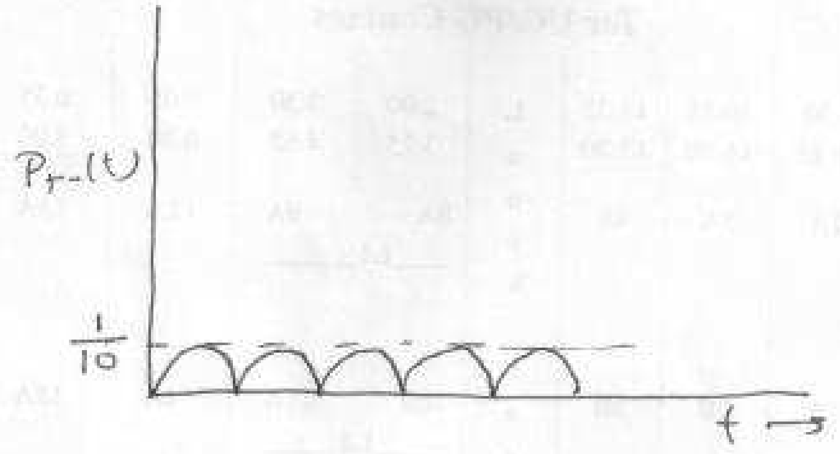
Now the probability of a spin flip at time $t$ $P_{+-}(t)$ is nothing but

$$\boxed{P_{+-}(t) = |\langle -|\psi(t)\rangle|^2 = \frac{\omega_1^2}{\omega_R^2}\sin^2\frac{\omega_R t}{2}}$$

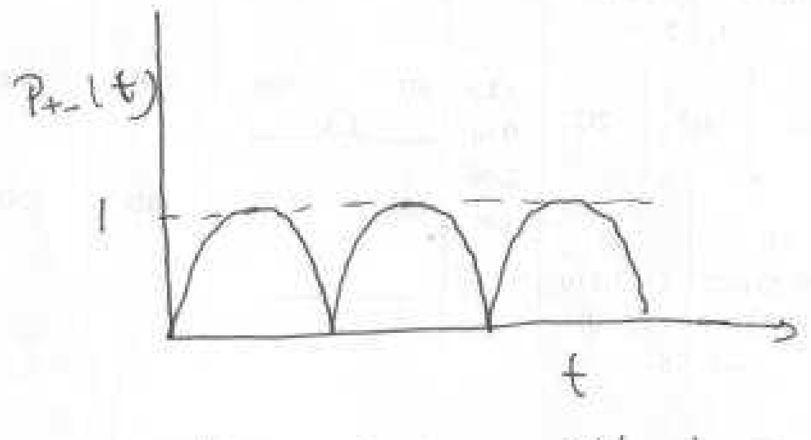
or

$$P_{+-}(t) = \frac{\omega_1^2}{\omega_1^2+\Delta\omega^2} \sin^2 \frac{\sqrt{\omega_1^2+\Delta\omega^2}\, t}{2}$$

for $\Delta\omega = 3\omega_1$ (off resonance)




at resonance $\Delta\omega = 0 \Rightarrow \omega = \omega_0$




Note that $P_{+-}(t)$ oscillates with time.
This is called Rabi oscillation.
At resonance the amplitude of oscillations is maximum possible ($=1$) and at time there is complete possibility of spin flip.