WKB method - continued

Two important formulae

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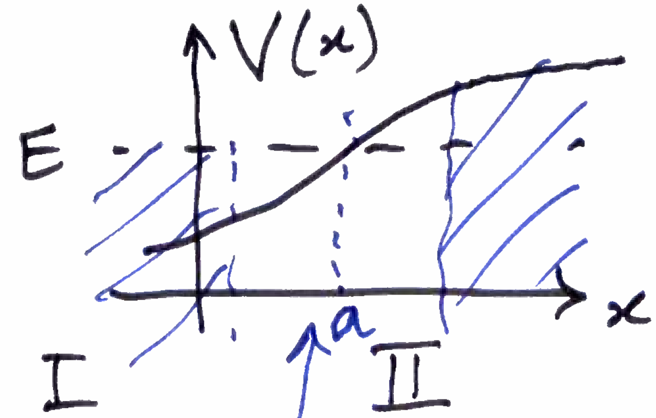
Matching the functions:

$$\psi(x,t) \simeq \frac{\text{const}}{[E-V(x)]^{1/4}} \exp\left\{ \pm \frac{i}{\hbar} \int_{x_0}^{x} dx' \sqrt{2m(E-V(x'))} - \frac{iEt}{\hbar} \right\}$$

The ansatz applicable also for $E - V(x) < 0$ again subject to $\frac{\hbar}{\sqrt{2m(V(x)-E)}} \ll \frac{2(V(x)-E)}{|dV/dx|}$

Thus we solve a different problem near the point $x = a$ where $E = V(x)$



WKB not valid

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + g(x-a)\psi = 0$$

$g = V'(x=a)$ taken to be > 0

Change variables $z = \left(\frac{2mg}{\hbar^2}\right)^{1/3}(x-a)$ s.t. $k^2 = \left(\frac{2mg}{\hbar^2}\right)^{2/3} z$

$$\frac{d^2\psi}{dz^2} - z\psi = 0$$

WKB requirement $|z|^{3/2} \gg \frac{1}{2}$

The solutions called Airy functions are Bessel functions of order 1/3.

Asymptotic forms:

$$Ai(z) \simeq \frac{1}{2}\frac{1}{\sqrt{\pi}}\frac{1}{z^{1/4}}e^{-\zeta}$$
$$Bi(z) \simeq \frac{1}{\sqrt{\pi}}\frac{1}{z^{1/4}}e^{\zeta}$$
for $z \to +\infty$

$$\zeta = \frac{2}{3}|z|^{3/2}$$

Whereas,

$$Ai(z) \simeq \frac{1}{\sqrt{\pi}}\frac{1}{|z|^{1/4}}\cos\left(\zeta - \frac{\pi}{4}\right)$$
$$Bi(z) \simeq -\frac{1}{\sqrt{\pi}}\frac{1}{|z|^{1/4}}\sin\left(\zeta - \frac{\pi}{4}\right)$$
$z \to -\infty$

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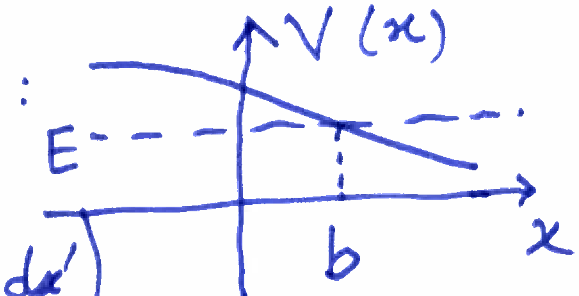
Connection formulae :

$$\frac{2A}{\sqrt{k(x)}}\cos\left(\int_x^a k(x')\,dx' - \frac{\pi}{4}\right) + \frac{(-B)}{\sqrt{k(x)}}\sin\left(\int_x^a k(x')\,dx' - \frac{\pi}{4}\right)$$

$$\longleftrightarrow \frac{A}{\sqrt{\chi(x)}}\exp\left\{-\int_a^x \chi(x')\,dx'\right\} + \frac{B}{\sqrt{\chi(x)}}\exp\left\{\int_a^x \chi(x')\,dx'\right\}$$

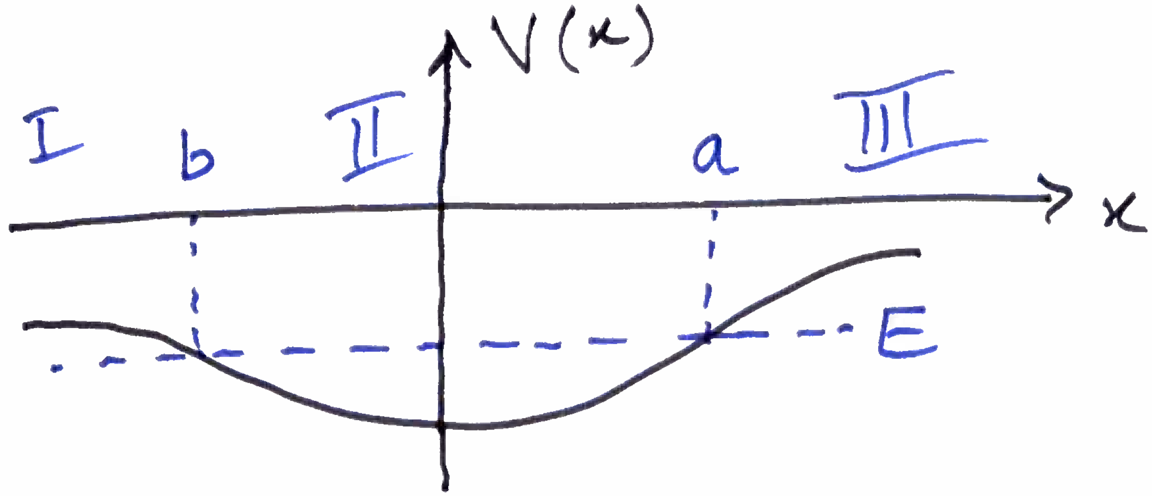
The inverse matching formula :



$$\frac{A}{\sqrt{\chi(x)}}\exp\left(-\int_x^b \chi(x')\,dx'\right) + \frac{B}{\sqrt{\chi(x)}}\exp\left(\int_x^b \chi(x')\,dx'\right)$$

$$\longleftrightarrow \frac{2A}{\sqrt{k(x)}}\cos\left(\int_b^x k(x')\,dx' - \frac{\pi}{4}\right) - \frac{B}{\sqrt{k(x)}}\sin\left(\int_b^x k(x')\,dx' - \frac{\pi}{4}\right)$$

Application to Bound states



a & b "classical turning points"

In region I we must have

$$\psi \simeq \frac{1}{\sqrt{\kappa}} \exp\left(-\int_x^b \kappa(x')\,dx'\right) \qquad x < b$$

Thus in region II,

$$\psi \simeq \frac{2}{\sqrt{k(x)}} \cos\left(\int_b^x k(x')\,dx' - \frac{\pi}{4}\right) \qquad b < x < a$$

Now, to match with region III

Re-write ψ_{II} in a different form,

$$\psi_{II} \simeq \frac{2}{\sqrt{k(x)}} \cos\left(\int_b^a k(x')dx' - \int_x^a k(x')dx' + \frac{\pi}{4} - \frac{\pi}{2} \right)$$

$$\simeq \frac{2}{\sqrt{k(x)}} \sin\left(\int_b^a k(x')dx' - \left(\int_x^a k(x')dx' - \frac{\pi}{4} \right) \right)$$

$$= -\frac{2}{\sqrt{k(x)}} \cos\left(\int_b^a k(x')dx' \right) \sin\left(\int_x^a k(x')dx' - \frac{\pi}{4} \right)$$

$$+ \frac{2}{\sqrt{k(x)}} \sin\left(\int_b^a k(x')dx' \right) \cos\left(\int_x^a k(x')dx' - \frac{\pi}{4} \right)$$

Since sin matches with growing exponential, this part of ψ_{II} must be zero, i.e.

$$\cos\left(\int_b^a k(x')\,dx'\right) = 0$$

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