Green Function
(for the Lippmann-Schwinger Equation)

$$(H_0 + V)(\psi^{(0)} + \psi^{(1)}) = (E^{(0)} + E^{(1)})(\psi^{(0)} + \psi^{(1)})$$

$$(E^{(0)} - H_0)\psi^{(1)} = V\psi^{(0)}$$

$$\psi^{(1)} = (E^{(0)} - H_0)^{-1} V \psi^{(0)}$$

... with due interpretation for inverting $(E^{(0)} - H_0)$ operator

L-S proposal:

$$\psi_n^{(1)+} = \lim_{\epsilon \to 0} \frac{1}{E^{(0)} - H_0 + i\epsilon} V \psi_n^{(0)}$$

... unlike bound state ~~problems~~ where $\sum_{k \neq n}$ suffices, here we need $i\epsilon$ because of continuum spectrum

Realise (obtain concrete expression for abstract operator relations) the L-S eqn. as a differential operator on the space of wave functions.

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2$$

and consider introducing the "master key" 2-point function / Green function / kernel s.t.

$$\left(E_n^{(0)} + \frac{\hbar^2}{2m}\nabla^2\right) G(\vec{x}-\vec{x}') = \delta^3(\vec{x}-\vec{x}')$$

Claim: Since On RHS, $\int d^3x' \, V(\vec{x}')\psi^{(0)}(\vec{x}')\delta^3(\vec{x}-\vec{x}')$ produces required RHS

hence expect $\int d^3x' \, G(\vec{x}-\vec{x}')V(\vec{x}')\psi^{(0)}(\vec{x}')$ is the reqd. $\psi^{(1)}$ on L.HS.

Let $G(\vec{x}-\vec{x}\,') = \int \frac{d^3k}{(2\pi)^3}\, e^{+i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\, \tilde{G}(\vec{k})$

[Assuming G to depend only on the difference of co-ord.s $\vec{x}-\vec{x}\,' \rightarrow$ i.e. assuming translation invariance]

And on RHS $\delta^3(\vec{x}-\vec{x}\,') = \int \frac{d^3k}{(2\pi)^3}\, e^{+i\vec{k}\cdot(\vec{x}-\vec{x}\,')}$

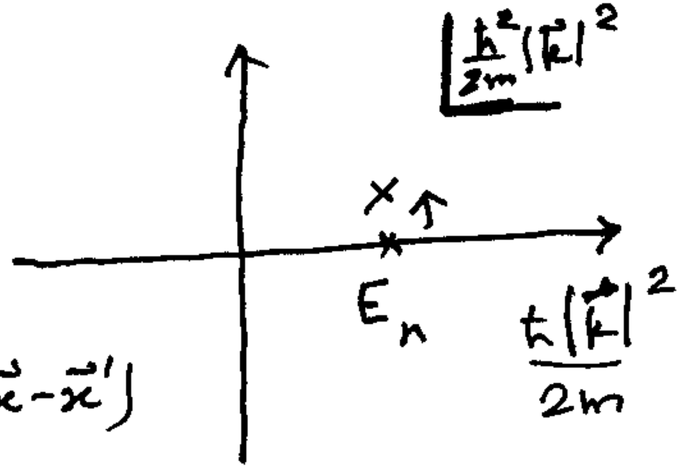
$$\left(E^{(0)} \mp \frac{\hbar^2|\vec{k}|^2}{2m}\right)\tilde{G}(\vec{k}) = 1$$

$$\therefore \quad \tilde{G}(\vec{k}) = \frac{1}{E_n^{(0)} - \frac{\hbar^2|\vec{k}|^2}{2m}}$$

i.e. $G(\vec{x}-\vec{x}\,') = \int \frac{d^3k}{(2\pi)^3} \times \exp\left(+i\vec{k}(\vec{x}-\vec{x})\right) \frac{1}{E^{(0)} - \frac{\hbar^2|\vec{k}|^2}{2m}}$

We recover the problem stated for abstract L-S eqn: the denominator has a pole when the d^3k integration reaches $\vec{k}$ values s.t. $\frac{\hbar^2|\vec{k}|^2}{2m} = E_n^{(0)}$

Prescription: Shift the $E_n^{(0)}$ to a complex value $E_n^{(0)} + i\epsilon$



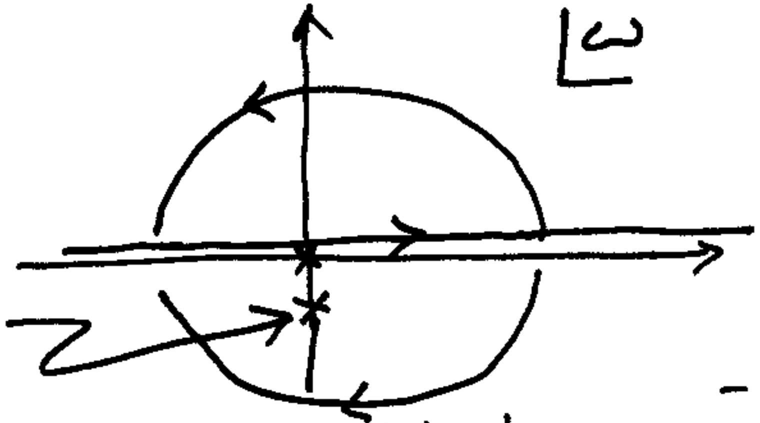
Why choose $+i\epsilon$?

$$G^{+}(\vec{x}, t, \vec{x}', t') = \int \frac{d^3k}{(2\pi)^3} \frac{e^{-iE_k(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')}}{E_n^{(0)} - \frac{\hbar^2|\vec{k}|^2}{2m}}$$

[Argument of the phase: $\omega t - kx$ ensures $x\uparrow$ as $t\uparrow$ to have the same phase]

Consider the simpler integral

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega + i\epsilon} e^{-i\omega\tau} \qquad \omega \to \frac{\hbar^2 |\vec{k}|^2}{2m}$$



$\omega = -i\epsilon$

$\text{Im}\,\omega > 0 \qquad \omega \equiv i|\omega| \qquad e^{-i(i|\omega|)\tau} = e^{|\omega|\tau}$

$\text{Im}\,\omega < 0 \qquad \omega \equiv -i|\omega| \qquad e^{-i(-i|\omega|)\tau} = e^{-|\omega|\tau}$

Want the pole to contribute when $\tau > 0$

Hence need $\text{Im}\,\omega < 0$ so that contrib. from semicircle is damped & vanishes as $\omega \to \infty$

Thus we expect our G.F to be

$$G(\vec{x}-\vec{x}') \sim \lim_{\epsilon\to 0}\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{E_n^{(0)} - \frac{\hbar^2|\vec{k}|^2}{2m} + i\epsilon}$$

Pre-view of next time :

Simplifying $\int G(\vec{x}-\vec{x}')\, V(\vec{x}')\, \psi^{(0)}(\vec{x}')\, d^3x'$

Simplify G.F. by assuming that $(\vec{x}-\vec{x}')$ is relatively small compared to $|\vec{x}|$. ~~or~~ $\cancel{|\vec{x}'|}$

$\vec{x}$ is over the whole space but $\vec{x}-\vec{x}'$ is over the range of the potential $V(\vec{x}')$