Proof of Path integral formula

Denote basis states as $|q\rangle$

...equivalent other basis $|p\rangle$

$$q_{op}|q\rangle = q|q\rangle \; ; \quad \langle q'|q\rangle = \delta(q-q')$$

$$p_{op}|p\rangle = p|p\rangle \; ; \quad \langle p'|p\rangle = \delta(p-p')$$

Thus wavefunction $\psi(q,t) = \langle q|\psi\, t\rangle$

Q. Kinematics:

$$\langle q | p \rangle = \frac{e^{iqp/\hbar}}{\sqrt{2\pi\hbar}}$$

This is equiv. to $p_{op} = -i\hbar \frac{d}{dq}$ Schrö.

or $[q_{op}, p_{op}] = i\hbar$ Heis.

Propose instantaneous basis states

$$q_H(t) \, |q\,t\rangle_D = q \, |q\,t\rangle_D$$

$\hookrightarrow$ "Dirac picture"

... why $|q\,t\rangle_D$?

$$|\psi t\rangle_S = e^{-iHt/\hbar}|\psi 0\rangle$$
$$\equiv e^{-iHt/\hbar}|\psi\rangle_H$$

Thus the relation of Schr. & Heis. pictures

$$\mathcal{O}_H(t) = e^{iHt/\hbar}\,\mathcal{O}_S\,e^{-iHt/\hbar}$$

Now note, $q_H(t)\,|q\,t\rangle_D = e^{iHt/\hbar}\,q_S\,e^{-iHt/\hbar}\,|q\,t\rangle_D$

$$= e^{iHt/\hbar}\,q_S\,\underbrace{e^{-iHt/\hbar}\,e^{iHt/\hbar}}_{1}\,|q\rangle$$

$$= q\,e^{iHt/\hbar}|q\rangle = \cancel{q|q\,t\rangle_D}\; q\,|q\,t\rangle_D$$

Now we recast the statement of time evolution using $|q\,t\rangle_D$:

$$\psi(q_f t_f) = \langle q_f | \psi t_f \rangle_S = {}_D\langle q_f t | \psi \rangle_H$$

$$\langle q_f | e^{-iHt_f/\hbar} | \psi 0 \rangle$$

Now we can use this to write

$$\psi(q_f t_f) = \int dq_i \; {}_D\langle q_f t_f | q_i t_i \rangle_D \, {}_D\langle q_i t_i | \psi \rangle_H$$

$t_f > t_i$

using $\mathbb{1} = \int dq_i \, | q_i t_i \rangle_D \, {}_D\langle q_i t_i |$

$$\psi(q_f t_f) = \int dq_i \, K(q_f t_f \, q_i t_i) \, \psi(q_i t_i)$$

$t_f > t_i$

with $K(q_f t_f ; q_i t_i) \equiv {}_D\langle q_f t_f | q_i t_i \rangle_D$

called the kernel or Green's Function

also can be called "propagator".

Now ${}_D\langle q_f t_f | q_i t_i \rangle_D = \langle q_f | e^{-iH(t_f - t_i)/\hbar} | q_i \rangle$

Can we calculate ${}_D\langle q \, t+\Delta t | q \, t \rangle_D$ for small Δt?

... can be compounded into final answer

...a useful different object

$$ {}_D\langle p\, t+\Delta t | q\, t\rangle_D = \langle p | e^{-iH\Delta t/\hbar} | q\rangle $$

$$ \simeq \langle p | \mathbb{1} - i\frac{H\Delta t}{\hbar} + O(\Delta t^2) .. | q \rangle $$

Normal ordering in H:

$H(q, p)$ written with all p on the left of all q's

Thus H above can be replaced by its value

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$$\langle p\, t+\Delta t | q, t\rangle \simeq \frac{e^{-iqp/\hbar}}{\sqrt{2\pi\hbar}} - \frac{i}{\hbar} H(p,q)\Delta t \frac{e^{-iqp/\hbar}}{\sqrt{2\pi\hbar}}$$

$$\sim \frac{e^{-iqp/\hbar}}{\sqrt{2\pi\hbar}}\left\{ 1 - \frac{i}{\hbar} H(p,q)\Delta t \ldots \right\}$$

$$\sim \frac{e^{-iqp/\hbar}}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} H(q,p)\Delta t} \quad \ldots \text{ if } \Delta t \text{ is small}$$

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Thus, consider

$$ {}_D\langle q_f t_f | q_i t_i \rangle_D = \int dp_f \, \underbrace{{}_D\langle q_f t_f | p_f t_f \rangle_D}_{\frac{e^{i q_f p_f/\hbar}}{\sqrt{2\pi\hbar}}} \, \underbrace{{}_D\langle p_f t_f | q_i t_i \rangle_D}_{\text{from previous pg. if } t_f - t_i \text{ is small}} $$

$$ \sim \int \frac{dp}{2\pi\hbar} \, e^{i(q_f p_f - q_i p_f)/\hbar - iH\Delta t/\hbar} $$

suggests $\rightarrow$

$$ \sim \int \frac{dp}{2\pi\hbar} \, e^{i p_f \frac{\Delta q}{\Delta t} \Delta t/\hbar - iH\Delta t/\hbar} $$

$$ \sim \int dp \; e^{i(p\dot{q} - H)\Delta t/\hbar} $$

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Introduce

$q_1, q_2 \dots q_N \; ; \; p_1, p_2 \dots p_N, p_{N+1} \equiv p_f$

at times $t_i < t_1 < t_2 \dots < t_N < t_f$

$$
{}_D\langle q_f t_f | q_i t_i \rangle_D = \int \langle q_f t_f | p_{N+1} t_f \rangle \langle p_{N+1} t_f | q_N t_N \rangle \times
$$

$$
\langle q_N t_N | p_N t_N \rangle \langle p_N t_N | q_{N-1} t_{N-1} \rangle \times
$$

$$
\dots
$$

$$
\times \langle q_1 t_1 | p_1 t_1 \rangle \langle p_1 t_1 | q_i t_i \rangle
$$

$$
\times \prod_1^N dq_i \prod_1^{N+1} dp_i
$$

As the number of slices $N \to \infty$, we get $\Delta t_N = t_N - t_{N-1} \to 0$ and our formula becomes

$$\lim_{N\to\infty} \int \frac{dp_{N+1}}{\sqrt{2\pi\hbar}} \prod_{j=1}^{N} \frac{dq_j\, dp_j}{2\pi\hbar} \exp\left\{\frac{i}{\hbar}\sum_j \left[p_{j+1}(q_{j+1}-q_j) - H(p_{j+1}, q_j)\Delta t_j \right]\right\}$$

Symbolically we write

$$\boxed{\int \underbrace{\mathcal{D}p\,\mathcal{D}q}_{} \exp\left\{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,\left(p\dot{q} - H(p,q)\right)\right\}}$$

integration over paths