Fredholm alternatives

Equation $Au = v$

Consider the solution in a general form

$$u_g = u_h + u_p$$

where $Au_h = 0$ ... homogeneous eqn.

and $Au_p = v$ $\quad u_p \to$ particular soln.

Alternatives: for homog. eqn.

a) $Au = 0$ has a ~~unique~~ non-trivial soln. $\det A = 0$

b) $Au = 0$ has no non-trivial soln. $\det A =$ $\det A \neq 0$

In the case b) suppose we can still solve ~~for~~ the particular eqn.

$$A u_p = v$$

Theorem : This possibility ($\det A = 0$ and obtaining well defined u_p) exists if and only if we use the fact that v is orthogonal to the space of homog. vectors

$$\langle u_h, v \rangle = \langle u_h, A u_p \rangle = \langle u_h, A u_g \rangle \quad \because A u_h = 0$$

$$= \langle A u_h, u_g \rangle \text{ (hermiticity of } A) = 0$$

We want to treat

$$A \equiv H_0 - \mathbb{1} E_n^{(0)}$$

$$v \equiv (\Delta E_n^{(\lambda)} - V) | n^{(0)} \rangle$$

and $u_p \equiv | \Delta n \rangle_\lambda$

Thus we carry out ~~solve~~ the iterative ~~eq~~ procedure keeping $|\Delta n\rangle_\lambda$ orthogonal to $| n^{(0)} \rangle$

Introduce a projection operator

$$P v = 0$$

$$A u_p = v = v - Pv = (\mathbb{1} - P) v$$

Suppose we now define K s.t.

$$A K = \mathbb{1} - P \qquad (\alpha)$$

K is effectively a properly defined inverse of A, restricted to space where it does not have zero eigenvalues.

Note: properties of projection operators

$P|b\rangle = 0$ for $\{|b\rangle\}$ basis vectors in a special subspace

$P^2 = P$. Typically $P = \mathbb{1} - \sum_b |b\rangle\langle b|$

check $(\mathbb{1} - \sum_{b'} |b'\rangle\langle b'|)|b\rangle = |b\rangle - \delta_{bb'}|b'\rangle$
$= 0$

Further we characterise K by demanding $K u_h = 0$ $\quad (\beta)$

Conditions (α) & (β) characterise the required "restricted" inverse of A.

Formally / symbolically,

$$\text{“}\, K \equiv \frac{\mathbb{1} - P}{A} \,\text{”} \equiv \sum_{k \neq u_h} \frac{|k\rangle\langle k|}{A}$$

Thus we go back to our problem and consider it in the form

$$|\Delta n\rangle_\lambda = \underbrace{C_0 |n^{(0)}\rangle}_{u_h} \underbrace{- K_n (E_n^{(1)} - V) |n^{(0)}\rangle}_{u_p}$$

$$= C_0 |n^{(0)}\rangle + K_n V |n^{(0)}\rangle$$

Choose $C_0 = 0$ since a first term $|n^{(0)}\rangle$ is present

And choose $K_n = \sum_{l \neq n} \frac{1}{E_n^{(0)} - E_l^{(0)}} |l^{(0)}\rangle\langle l^{(0)}|$

i.e., K_n is characterised by

$$\left(-E_n^{(0)} + H_0\right) K_n = \mathbb{1} - |n^{(0)}\rangle\langle n^{(0)}|$$

... to be contined

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